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A. Maitra
R. Purves
W. Sudderth

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APPROXIMATION THEOREMS FOR GAMBLING PROBLEMS
AND STOCHASTIC GAMES

A. Maitra
Santa Fe Institute

R. Purves
University of California, Berkeley

W. Sudderth
University of Minnesota

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Abstract

A two-person, zero-sum stochastic game with a countable state space $X$ and finite action sets is known to have a value $V(C)(V(0))$ if the payoff function is the indicator of a closed subset $C$ (open subset 0) of the product space $X \times X \times \ldots$. It is shown here that if the payoff function is the indicator of a Borel subset $E$ of $X \times X \times \ldots$, then the game has a value, if and only if, for every $\epsilon > 0$, there exist a closed set $C$ and an open set 0 such that $C \subseteq E \subseteq 0$ and $V(0) - V(C) < \epsilon$. The proof relies on an approximation result for gambling problems with a countable state space and the proof of this result depends, in turn, on an application of Choquet's Capacitability Theorem.

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1. INTRODUCTION

Let \( X \) be a countable, nonempty set of states, and let \( A \) and \( B \) be finite, nonempty sets of actions for players I and II, respectively. Let \( q \) be a function which assigns to each triple \((x, a, b)\) in \( X \times A \times B \) a probability distribution \( q(\cdot \mid x, a, b) \) on \( X \). Set \( Z = A \times B \times X \) and let \( N = \{1, 2, \ldots\} \). Give \( Z^N \), the space of infinite sequences \((z_1, z_2, \ldots)\) of elements of \( Z \), the product topology when \( Z \) is endowed with the discrete topology. Let \( \varphi \) be a bounded, Borel measurable function on \( Z^N \).

The game \( G_x(\varphi) \) starts at some initial state \( x \). Player I chooses an action \( a_1 \in A \) and, simultaneously, player II chooses \( b_1 \in B \). The next state \( x_1 \) has distribution \( q(\cdot \mid x, a_1, b_1) \) and is announced to the players along with their chosen actions. The procedure is iterated so as to generate a sequence \( z_1 = (a_1, b_1, x_1), z_2 = (a_2, b_2, x_2), \ldots \). The payoff from player II to player I is \( \varphi(z_1, z_2, \ldots) \).

A strategy \( \sigma \) for player I is a sequence \( \sigma_0, \sigma_1, \ldots \), where \( \sigma_0 \in P(A) \) and, for \( n \geq 1 \), \( \sigma_n \) is a mapping from \( Z^n \) into \( P(A) \), where \( P(A) \) is the set of probability measures on \( A \). A strategy \( \tau \) for player II is defined analogously with \( P(A) \) replaced by \( P(B) \), the set of probability measures on \( B \). Strategies \( \sigma \) and \( \tau \) together with an initial state \( x \) determine a probability measure \( P_{x, \sigma, \tau} = P_{x, \sigma_1, \tau} \) on the Borel subsets of \( Z^N \). (The initial state will usually be clear from the context and we will usually suppress it.) When I uses \( \sigma \) and II uses \( \tau \), the expected payoff from II to I is \( E_{\sigma, \tau} \varphi = E_x \varphi = \int \varphi dP_{x, \sigma, \tau} \).

The upper value of the game \( G_x(\varphi) \) is defined by

\[
\tilde{V}_x(\varphi) = \inf_{\sigma} \sup_{\tau} E_{x, \sigma, \tau} \varphi.
\]

Analogously, the lower value of the game \( G_x(\varphi) \) is defined by

\[
V_x(\varphi) = \sup_{\sigma} \inf_{\tau} E_{x, \sigma, \tau} \varphi.
\]

The game \( G_x(\varphi) \) is said to have a value if \( \tilde{V}_x(\varphi) = V_x(\varphi) \), in which case the value will be denoted by \( V_x(\varphi) \).

If in the above formulation of the game \( G_x(\varphi) \) we omit the state space \( X \) and the law of motion \( q \), we get the formulation of Blackwell [1]. He proved that if \( \varphi \) is the indicator function of a \( G_\delta \) subset of \( Z^N \), then the game \( G_x(\varphi) \) has a value. This result was extended by Orkin [6] to the case where \( \varphi \) is the indicator function of a set belonging to the field generated by the \( G_\delta \) subsets of \( Z^N \). (Both results carry over without effort to the present, more general, formulation.) For functions which are not \( 0 - 1 \) valued, the best result know to date is that if \( \{\varphi \geq c\} \) is a \( G_\delta \) for every real number \( c \), then \( G_x(\varphi) \) has a value. We will deduce this result in an appendix to this paper as an easy consequence of the main result of Maitra and Sudderth [4].

This is where matters stand today with respect to the game \( G_x(\varphi) \). In the present article, we will consider only win-lose games as Blackwell and Orkin did. While we have not been able to extend Orkin's
result to a wider class—indeed, we do not know if \( G_\varphi(\varphi) \) has a value when \( \varphi \) is the indicator function of a \( G \) subset of \( Z^N \), the main result of this article will give a necessary and sufficient condition for a win-lose game to have a value.

Write \( G_\varphi(E), V_\varphi(E), V_\varphi(E) \) for \( G_\varphi(\varphi), V_\varphi(\varphi), V_\varphi(\varphi) \) in case \( \varphi \) is the indicator function of a Borel subset \( E \) of \( Z^N \). The main result of the paper can now be stated.

**Theorem 1.1**

Let \( E \) be a Borel subset of \( Z^N \). In order that the game \( G_\varphi(E) \) have a value, it is necessary and sufficient that

\[
\inf V_\varphi(0) = \sup V_\varphi(C),
\]

where the inf is taken over all open subsets of \( Z^N \) containing \( E \) and the sup is taken over all closed subsets \( C \) of \( Z^N \) contained in \( E \).

The proof of Theorem 1.1 will be based on a new approximation theorem in the Dubins-Savage theory of gambling [3]. The next section will contain the basic definitions and concepts of the theory of gambling. In section 3, we will establish the approximation theorem and section 4 will be devoted to the proof of Theorem 1.1.

### 2. DEFINITIONS AND NOTATION FROM GAMBLING THEORY

Let \( X \) be a nonempty, countable set and let \( P(X) \) be the set of probability measures defined on the power-set of \( X \). A *gambling house* \( \Gamma \) on \( X \) is a mapping which assigns to each \( x \in X \) a nonempty set \( \Gamma(x) \subseteq P(X) \). A player in the house \( \Gamma \) starts at some initial state \( x \) and chooses a *strategy* \( \sigma \) available at \( x \), which means a sequence \( \sigma = (\sigma_0, \sigma_1, \ldots) \), where \( \sigma_0 \in \Gamma(x) \) and, for \( n = 1, 2, \ldots, \sigma_n \) is a mapping from \( X^n \) to \( P(X) \) such that \( \sigma_n(x_1, x_2, \ldots, x_n) \subseteq \Gamma(x_n) \) for every \( (x_1, x_2, \ldots, x_n) \in X^n \). Denote by \( H \) the set of infinite sequences \( (x_1, x_2, \ldots) \) of elements of \( X \) and give \( H \) the product topology when \( X \) is endowed with this discrete topology. Every strategy \( \sigma \) at \( x \) determines a probability measure on the Borel subsets of \( H \). This probability measure, also denoted by \( \sigma \), can be regarded as the distribution of the coordinate process \( h_1, h_2, \ldots \), where \( h_1 \) has distribution \( \sigma_0 \) and \( h_{n+1} \) has conditional distribution \( \sigma_n(x_1, x_2, \ldots, x_n) \) given that \( h_1 = x_1, h_2 = x_2, \ldots, h_n = x_n \). For \( x \in X \), let \( \Sigma(x) \) be the collection of all strategies at \( x \). In the sequel, we shall frequently regard \( \Sigma(x) \) as a set of probability measures on \( H \), viz. the probability measures induced by strategies at \( x \).
Let $E$ be an analytic subset of $H$. The **optimal reward operator** $M$ assigns to each such set $E$ the function $M(E)$ on $X$ defined by

$$M(E)(x) = \sup\{\sigma(E) : \sigma \in \Sigma(x)\}.$$  

In order to formulate the approximation result, we define a related operator $\tilde{M}$, which assigns to every subset $E$ (not just analytic subsets) of $H$ the function $\tilde{M}(E)$ on $X$ defined by

$$\tilde{M}(E)(x) = \inf\{M(0)(x) : 0 \text{ is open and } 0 \supseteq E\}.$$  

The fundamental approximation theorem is then

**Theorem 2.1**

If $E$ is an analytic subset of $H$, then $M(E) = \tilde{M}(E)$.

We defer the proof of Theorem 2.1 to the next section. We will now continue with our definitions.

For $h, h' \in H$ and for a natural number $n$, we write $h \equiv_n h'$ if $h_i = h'_i$, $i = 1, 2, \ldots, n$. A mapping $t$ from $H$ to $N \cup \{\infty\}$ is called a stopping time if

$$t(h) = n \in N \text{ and } h' \equiv_n h \text{ imply } t(h') = n.$$  

If $t$ is a stopping time, then $\{t < \infty\}$ is an open subset of $H$. Conversely, if $0$ is an open subset of $H$, then there is a stopping time $t$ such that $0 = \{t < \infty\}$.

Suppose that $\sigma$ is a strategy at $x$ and $p \in X^m$. Define a strategy $\sigma[p]$ at $l[p]$, the last coordinate of $p$, as follows:

$$(\sigma[p])_0 = \sigma_m(p)$$

and, for $n \geq 1$,

$$(\sigma[p])_n(x_1, x_2, \ldots, x_n) = \sigma_{m+n}(px_1, x_2, \ldots, x_n),$$

where $px_1, x_2, \ldots, x_n$ is the element of $X^{m+n}$ obtained by catenating $p$ and $(x_1, x_2, \ldots, x_n)$. It is easy to verify that the measures (induced by) $\sigma[p], p \in X^m$ are a version of the conditional $\sigma$-distribution of $(h_{m+1}, h_{m+2}, \ldots)$ given $(h_1, h_2, \ldots, h_m) = p$. If $t$ is a stopping time and $t(h) < \infty$, we write $h_t = h_t(h)$ for $h_t(h), p_t = p_t(h)$ for $(h_1, h_2, \ldots, h_t)$ and $\sigma[p_t] = \sigma[p_t(h)]$ for $\sigma[p_t(h)]$.

If $B \subseteq H$ and $p \in X^m$, then $Bp$ will denote the set of $hcH$ such that $ph \in B$, where $ph$ is the element of $H$ obtained by catenating $p$ and $h$. Similarly, if $t$ is a stopping time and $t(h) < \infty$, write $Bp_t = (Bp_t)(h)$ for $Bp_t(h)$.
We will prove Theorem 2.1 by showing that, for every $x \in X$, $\bar{M}(\cdot)(x)$ is a capacity and then appealing to Choquet's Capacitability Theorem. Recall that a real-valued function $J$ on the power-set of $H$ is a capacity if

a. $J$ is monotone, i.e., $E_1 \subseteq E_2 \subseteq H$ implies $J(E_1) \leq J(E_2)$;

b. $J$ has the "going up" property, i.e., $E_n \uparrow E \subseteq H$ implies $J(E) = \lim_n J(E_n)$, and

c. $J$ is "right-continuous on compacts," i.e.,

$$J(K) = \inf\{J(0): 0 \text{ is open and } 0 \supseteq K\}$$

for every compact $K \subseteq H$.

3. PROOF OF THEOREM 2.1

To commence the proof, note that for fixed $x \in X$, the set function $\bar{M}(\cdot)(x)$ satisfies clauses (a) and (c) in the definition of a capacity. However, that $\bar{M}(\cdot)(x)$ satisfies clause (b) as well turns out to be quite non-trivial. This will be proved through a sequence of lemmas, the penultimate of which will actually establish Theorem 2.1 in the special case that $E$ is a $G_{\delta}$ subset of $H$.

Lemma 3.1

Let $t$ be a stopping time and let $E \subseteq \{t < \infty\}$. Then, for every $x \in X$,

$$\bar{M}(E)(x) \leq \sup_{\sigma \in \sum(x)} \int_{\{t < \infty\}} \bar{M}(Ep(h))(h) d\sigma. \tag{3.0}$$

Proof.

Fix $x \in X$ and $\epsilon > 0$. For each $p \in U_{m}X^n$, choose an open set $0(p) \supseteq Ep$ such that

$$M(0(p))(l(p)) \leq \bar{M}(Ep)(l(p)) + \epsilon,$$

where $l(p)$ is the last coordinate of $p$. Define

$$0 = \{h \in H: (\exists n)(l(h) = n \text{ and } (h_{n+1}, h_{n+2}, \ldots) \in 0(p_n(h)))\}$$

where $p_n(h) = (h_1, h_2, \ldots, h_n)$.

Plainly, $0$ is open and $E \subseteq 0 \subseteq \{t < \infty\}$.
Hence, for any \( \sigma \in \Sigma_0(x) \),

\[
\sigma(0) = \int_{\{t<\infty\}} \sigma[p_t](0)p_t(h) d\sigma \\
= \int_{\{t<\infty\}} \sigma[p_t](0(p_t(h))) d\sigma \\
\leq \int_{\{t<\infty\}} M(0)p_t(h) d\sigma \\
\leq \int_{\{t<\infty\}} \bar{M}(E|p_t)(h_t) d\sigma + \epsilon.
\]

Take the sup over all \( \sigma \in \Sigma_0(x) \) to obtain

\[
M(0)(x) \leq \sup_{\sigma \in \Sigma_0(x)} \int_{\{t<\infty\}} \bar{M}(E|p_t)(h_t) d\sigma + \epsilon.
\]

Since 0 is open and contains \( E \), \( \bar{M}(E)(x) \leq M(0)(x) \) and the left-hand side of Eq. (3.0) is less than or equal to the right-hand side.

Lemma 3.2

Suppose \( G \subseteq H \) is a \( G_f \) set. Then \( \bar{M}(G) = M(G) \).

Proof.

Write \( G \) as the countable intersection of non-increasing open sets \( V_n, n = 1, 2, \ldots \). Choose stopping times \( t_n, n \geq 1 \), such that for each \( n \geq 1 \), \( V_n = \{ t_n < \infty \} \) and \( t_n < t_{n+1} \) on \( \{ t_{n+1} < \infty \} \).

Fix \( x_0 \in X \) and \( \epsilon > 0 \). To prove the lemma, it will suffice to produce a strategy \( \sigma \in \Sigma(x_0) \) such that

\[
\sigma(G) \geq \bar{M}(G)(x_0) - \epsilon.
\] (3.1)

We construct \( \sigma \) by repeated applications of Lemma 3.1. Choose \( \sigma^0 \in \Sigma(x_0) \) by Lemma 3.1 such that

\[
\int_{\{t<\infty\}} \bar{M}(G|p_t(h)) d\sigma^0(h) \geq \bar{M}(G)(x_0) - \frac{\epsilon}{2}.
\] (3.2)

Next suppose that \( t_n(h) < \infty \). Define the stopping time \( t_{n+1}[p_{t_n}(h)] \) by the formula

\[
t_{n+1}[p_{t_n}(h)](h') = t_{n+1}(p_{t_n}(h)h') - t_n(h)
\]

and apply Lemma 3.1 to the set \( G|p_{t_n}(h) \) and the stopping time \( t_{n+1}[p_{t_n}(h)] \) to obtain \( \sigma^n(p_{t_n}(h)) \epsilon \Sigma(h_{t_n}) \) such that

\[
\int_{\{t_{n+1}[p_{t_n}(h)] < \infty\}} \bar{M}(G|p_{t_n}(h)p_{t_{n+1}[p_{t_n}(h)]}(h')(h'_{t_{n+1}[p_{t_n}(h)]})) d\sigma^n(p_{t_n}(h))(dh') \geq \bar{M}(G|p_{t_n}(h))(h_{t_n}) - \frac{\epsilon}{2^{n+1}}.
\] (3.3)
We can now define $\sigma$ as follows:

$$\sigma_0 = \sigma^0$$

and, for $m \geq 1$,

$$\sigma_m(x_1, x_2, \ldots, x_m) = \sigma^0_m(x_1, x_2, \ldots, x_m) \text{ if } m < t_1(x_1, x_2, \ldots, x_m \ldots)$$

$$= \sigma^m(p_{t_n}(x_1, x_2, \ldots, x_m, \ldots) - t_n(x_1, x_2, \ldots, x_m, \ldots) + 1, \ldots, x_m)$$

if $t_n(x_1, x_2, \ldots, x_m, \ldots) \leq m < t_{n+1}(x_1, x_2, \ldots, x_m, \ldots)$.

Plainly, $\sigma \in \Sigma(x_0)$. We will verify Eq. (3.1). For $n \geq 1$, define

$$Y_n(h) = \bar{M}(G_{t_n}(h))(h_{t_n}), \text{ if } h \in V_n$$

$$= 0, \text{ otherwise.}$$

In the calculations below, expectations and conditional expectations are with respect to $\sigma$. According to Eq. (3.2),

$$E(Y_1) \geq \bar{M}(G)(x_0) - \frac{\varepsilon}{2}.$$ 

From Eq. (3.3), we have:

$$E(Y_{n+1} \mid F_{t_n}) \geq Y_n - \frac{\varepsilon}{2^{n+1}} \quad a.s.(\sigma),$$

where $F_{t_n}$ is the pre-$t_n$ sigma-field. Taking expectations, we obtain

$$E(Y_{n+1}) \geq E(Y_n) - \frac{\varepsilon}{2^{n+1}},$$

so that

$$E(Y_{n+1}) \geq E(Y_1) - \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \ldots + \frac{\varepsilon}{2^{n+1}}\right)$$

$$\geq \bar{M}(G)(x_0) - \varepsilon, n = 1, 2, \ldots.$$

Hence

$$\limsup_n E(Y_n) \geq \bar{M}(G)(x_0) - \varepsilon.$$

Since $\{Y_n > 0\} \subseteq V_n$ and $0 \leq Y_n \leq 1$, it follows that $E(Y_n) \leq \sigma(V_n)$. Consequently,

$$\sigma(G) = \lim_{n} \sigma(V_n)$$

$$\geq \limsup_n E(Y_n)$$

$$\geq \bar{M}(G)(x_0) - \varepsilon.$$

Lemma 3.3

Let $E$ be a $G_{t_\sigma}$ subset of $H$. If $M(E)(x) = 0$, then $\bar{M}(E)(x) = 0.$
Proof.

Write $E = U_{n \geq 1} E_n$, where the sets $E_n$ are $G_\delta$ subsets of $H$. Plainly, for each $n \geq 1, M(E_n)(x) = 0$ and, so by Lemma 3.2, $\bar{M}(E_n)(x) = 0$. Fix $\epsilon > 0$. For each $n \geq 1$, choose an open set $O_n \supseteq E_n$ such that $M(O_n)(x) < \epsilon/2^n$. Set $O = U_{n \geq 1} O_n$. Then $O$ is open and contains $E$. Now $M(\cdot)(x)$ is easily seen to be countably subadditive and, consequently,

$$\bar{M}(E)(x) \leq M(O)(x) \leq \sum_{n} M(O_n)(x) < \epsilon.$$

For the next two lemmas, fix $\epsilon > 0$ and a Borel (or even universally measurable) subset $E$ of $H$.

Define a stopping time $t_\epsilon$ as follows:

$$t_\epsilon(h) = \inf\{n \geq 1: M(Ep_n(h))(h_n) > 1 - \epsilon\}, \tag{3.4}$$

where $p_n(h) = (h_1, h_2, \ldots, h_n)$ and we use the convention that $\inf(\emptyset) = +\infty$.

Lemma 3.4

For every $x \in X$,

$$M(E \cap \{t_\epsilon = \infty\})(x) = 0.$$

Proof.

Towards a contradiction, assume there is a $\sigma \epsilon \sum(x)$ such that $\sigma(E \cap \{t_\epsilon = \infty\}) > 0$. By the Lèvy 0 − 1 law, there exist $h \in E \cap \{t_\epsilon = \infty\}$ and $m \geq 1$ such that

$$\sigma[p_m(h)]((E \cap \{t_\epsilon = \infty\})p_m(h)) > 1 - \epsilon,$$

and hence,

$$\sigma[p_m(h)](Ep_m(h)) > 1 - \epsilon,$$

so that $t_\epsilon(h) \leq m$, contradicting the fact that $t_\epsilon(h) = \infty$.

Lemma 3.5

For every $x \in X$,

$$M(\{t_\epsilon < \infty\})(x) \leq M(E \cap \{t_\epsilon < \infty\})(x) + 2\epsilon.$$
Fix $x \in X$. Choose $\sigma_t \epsilon \sum(x)$ such that

$$\sigma_t^0(\{t < \infty\}) \geq M(\{t < \infty\})(x) - \frac{\epsilon}{2}.$$ 

For every $p \epsilon U_{m \geq 1} X^m$, choose $\sigma(p) \epsilon \sum(l(p))$ such that

$$\sigma(p)(Ep) \geq M(\{t = \infty\})(l(p)) - \frac{\epsilon}{2}.$$ 

Define a new strategy $\sigma$ as follows:

$$\sigma_0 = \sigma_t^0$$

and, for $m \geq 1$,

$$\sigma_m(x_1, x_2, \ldots, x_m) = \sigma_t^0(x_1, x_2, \ldots, x_m) \text{ if } m < t_\epsilon(x_1, x_2, \ldots, x_m)$$

$$= \sigma(p_t(x_1, x_2, \ldots, x_m)) \text{ if } t_\epsilon(x_1, x_2, \ldots, x_m) \leq m$$

Plainly, $\sigma \epsilon \sum(x)$. Now calculate as follows:

$$M(E \cap \{t_\epsilon < \infty\})(x) \geq \sigma(E \cap \{t_\epsilon < \infty\})$$

$$= \int_{\{t_\epsilon < \infty\}} \sigma[p_t(h)](Ep_t(h)) \; d\sigma_t^0(h)$$

$$\geq \int_{\{t_\epsilon < \infty\}} M(\{t < \infty\})(h_\epsilon) \; d\sigma_t^0(h) - \frac{\epsilon}{2}$$

$$\geq (1 - \epsilon)\sigma_t^0(\{t_\epsilon < \infty\}) - \frac{\epsilon}{2}$$

$$\geq M(\{t_\epsilon < \infty\})(x) - 2\epsilon.$$

**Lemma 3.6**

If $E$ is a $G_{\delta_{\sigma}}$ subset of $H$, then $\bar{M}(E) = M(E)$.

**Proof.**

Fix $\epsilon > 0$ and $x \in X$. Let $t_\epsilon$ be defined by Eq. (3.4). Note that $E \cap \{t_\epsilon = \infty\}$ is a $G_{\delta_{\sigma}}$ subset of $H$. So, by Lemmas 3.3 and 3.4, $\bar{M}(E \cap \{t_\epsilon = \infty\})(x) = 0$. Hence, there is an open set $0'$ containing $E \cap \{t_\epsilon = \infty\}$ such that $M(0')(x) < \epsilon$. Set $0 = \{t_\epsilon < \infty\} \cup 0'$. Then $0$ is open and contains $E$. Furthermore, by Lemma 3.5,

$$\bar{M}(E)(x) \leq M(0)(x)$$

$$\leq M(\{t_\epsilon < \infty\})(x) + M(0')(x)$$

$$\leq M(E \cap \{t_\epsilon < \infty\})(x) + 3\epsilon$$

$$\leq M(E)(x) + 3\epsilon.$$ 

Since $\epsilon$ is arbitrary, this proves that $\bar{M}(E)(x) \leq M(E)(x)$. 

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Lemma 3.7

For every $x \in X$, $\tilde{M}(\cdot)(x)$ has the "going up" property.

Proof.

Let $E_n, n \geq 1$, be a non-decreasing sequence of subsets of $H$. Set $E = U_{n\geq 1} E_n$. Suppose that 

$$\lim_n \tilde{M}(E_n)(x) < a.$$ 

For each $n \geq 1$, choose an open set $0_n$ containing $E_n$ such that $M(0_n)(x) < a$. Let $F = U_{n\geq 1} \cap_{k\geq n} 0_k$.

Then $F$ is a $G_{\delta\sigma}$ subset of $H$ and contains $E$. Hence,

$$\tilde{M}(E)(x) \leq \tilde{M}(F)(x)$$

$$= M(F)(x)$$

$$= \sup_n M(\bigcap_{k\geq n} 0_k)(x)$$

$$\leq \sup_n M(0_n)(x)$$

$$\leq a,$$

where the first equality is by virtue of Lemma 3.6 and the second equality is by the easily verified fact that $M(\cdot)(x)$ has the "going up" property along Borel sets. It follows that $\tilde{M}(E)(x) \leq \lim_n (E_n)(x)$. Since the reverse inequality is obvious, the proof is complete.

We have, therefore, proven the following

Theorem 3.8

$\tilde{M}(\cdot)(x)$ is a capacity on $H$ for every fixed $x \in X$.

The proof of Theorem 2.1 is now easy. Let $E$ be an analytic subset of $H$. According to the Capacitability Theorem [2], for any $x \in X$, we have:

$$\tilde{M}(E)(x) = \sup \{ \tilde{M}(K)(x) : K \text{ is compact and } K \subseteq E \} .$$

On the other hand, it is easy to check that

$$M(E)(x) = \sup \{ M(K)(x) : K \text{ is compact and } K \subseteq E \} .$$

But by Lemma 3.2, $\tilde{M}(K)(x) = M(K)(x)$ for every compact $K$. Consequently, $\tilde{M}(E)(x) = M(E)(x)$ and we are done.
4. PROOF OF THEOREM 1.1

We return to the game theory problem formulated in section 1. Theorem 1.1 will follow trivially from the next result. The notation is as in section 1.

Theorem 4.1

Let $E$ be a Borel subset of $\mathbb{Z}^N$. Then, for every $x \in X$,

(i) $\bar{V}_x(E) = \inf\{V_x(0): 0$ is open and $0 \supseteq E \}$

and

(ii) $V_x(E) = \sup\{V_x(C): C$ is closed and $C \subseteq E \}$.

Proof.

To prove (i), fix a strategy $\tau$ for player II in the game. For $x \in X$, $\mu \in P(A)$ and $\alpha \in P(B)$, define a probability measure $m = m(x, \mu, \alpha)$ on the power-set of $\mathbb{Z}$ by setting

$$m(\{z\}) = \mu(\{a\})\alpha(\{b\})q(\{x_1\} | x, a, b)$$

where $z = (a, b, x_1)$.

Now define a gambling problem, as in sections 2 and 3, with state space $X^*$ and gambling house $\Gamma^*$ as follows:

$$X^* = X \cup \mathbb{Z} \cup \mathbb{Z}^2 \cup \ldots,$$

where the union above is a disjoint union. The gambling house $\Gamma^*$ on $X^*$ is defined by setting

$$\Gamma^*(x) = \{m(x, \mu, \tau_0): \mu \in P(A)\} \text{ if } x \in X;$$

$$\Gamma^*(p) = \{m((x_n), \mu, \tau_n(p))f_{\mu}^{-1}: \mu \in P(A)\} \text{ if } p \in \mathbb{Z}^n, n \geq 1,$$

where $(a_n, b_n, x_n)$ is the $n$th coordinate of $p$ and $f_{\mu}: \mathbb{Z} \to \mathbb{Z}^{n+1}$ is defined by $f_{\mu}(z) = pz$.

Set $H^* = X^{*N}$ and give $H^*$ the product topology when $X^*$ is endowed with the discrete topology.

Define $\psi: \mathbb{Z}^N \to H^*$ by

$$\psi((z_1, z_2, \ldots)) = ((x_1), (z_1, z_2), (z_1, z_2, z_3), \ldots).$$

Plainly, $\psi$ is a homeomorphism from $\mathbb{Z}^N$ into $H^*$.

The key observation linking player I's strategies in the game and the gambler's strategies in the gambling problem is the following:
For every Borel subset $F$ of $H^*$ and every $x \in X$,

$$\sup_{\sigma} P_{x, \sigma, \tau}(\psi^{-1}(F)) = \sup_{\sigma \in \sum^* \{x\}} \sigma^*(F). \quad (4.1)$$

where the sup on the left is over all strategies of player I in the game and on the right $\sum^* \{x\}$ is the set of all strategies available at $x$ in the gambling problem.

We omit the proof of (4.1). A proof is easily written down by imitating the proof of Lemma 6.1 in [5].

Let $E^* = \psi(E)$. Observe that $E^*$ is a Borel subset of $H^*$ and $\psi^{-1}(E^*) = E$. By (4.1) and Theorem 2.1, we have for every $x \in X$,

$$\sup_{\sigma} P_{x, \sigma, \tau}(E) = \sup_{\sigma \in \sum^* \{x\}} \sigma^*(E^*)$$

$$= \inf\{ \sup_{\sigma \in \sum^* \{x\}} \sigma^*(0^*): 0^* \text{ is open in } H^* \text{ and } 0^* \ni E^* \}$$

$$= \inf\{ \sup_{\sigma} P_{x, \sigma, \tau}(\psi^{-1}(0^*)): 0^* \text{ is open in } H^* \text{ and } 0^* \ni E^* \}$$

$$\geq \inf\{ \sup_{\sigma} P_{x, \sigma, \tau}(0): 0 \text{ is open in } Z^N \text{ and } 0 \ni E \}$$

and, hence,

$$\sup_{\sigma} P_{x, \sigma, \tau}(E) = \inf\{ \sup_{\sigma} P_{x, \sigma, \tau}(0): 0 \text{ is open in } Z^N \text{ and } 0 \ni E \}. \quad \text{(ii)}$$

This equality holds for every strategy $\tau$ of player II. Now take the inf over all strategies $\tau$ of II.

We obtain:

$$\bar{V}_x(E) = \inf_{\tau} \inf\{ \sup_{\sigma} P_{x, \sigma, \tau}(0): 0 \text{ is open and } 0 \ni E \}$$

$$= \inf\{ \inf_{\tau} \sup_{\sigma} P_{x, \sigma, \tau}(0): 0 \text{ is open and } 0 \ni E \}$$

$$= \inf\{ \bar{V}_x(0): 0 \text{ is open and } 0 \ni E \}. \quad \text{(ii)}$$

This proves (i).

In order to establish (ii), we interchange the roles of players I and II, so II becomes the maximizing player, i.e., she wants to maximize the probability of entering $E^*$. As before, $\sigma(\tau)$ will denote a strategy of I(II). Then the result established in (i) becomes

$$\inf_{\sigma} \sup_{\tau} P_{x, \sigma, \tau}(E^*) = \inf\{ \inf_{\sigma} \sup_{\tau} P_{x, \sigma, \tau}(0): 0 \text{ is open and } 0 \ni E^* \}$$

The left side is easily seen to be equal to

$$1 - \sup_{\sigma} P_{x, \sigma, \tau}(E),$$

while the right side is equal to

$$1 - \inf\{ \sup_{\sigma} P_{x, \sigma, \tau}(C): C \text{ is closed and } C \subseteq E \}.$$

This proves (ii).
5. APPENDIX

As promised in the Introduction, we will now prove

Theorem 5.1

Suppose \( \phi \) is a bounded function on \( Z^N \) such that \( \{ \phi \geq c \} \) is a \( G_\delta \) subset of \( Z^N \) for every real \( c \). Then, for every \( x \in X \), the game \( G_x(\phi) \) has a value.

Proof.

Without loss of generality, assume that \( 0 \leq \phi \leq 1 \). Fix \( \epsilon > 0 \). Let \( n(\epsilon) \) be the last integer \( n \) such that \( n \epsilon \geq 1 \) and, for \( n = 1, 2, \ldots, n(\epsilon) \), define

\[
G^n = \{(z_1, z_2, \ldots) \in Z^N : \phi((z_1, z_2, \ldots)) \geq n \epsilon \}.
\]

Then set

\[
\tilde{\phi} = \epsilon [G^1 + G^2 + \ldots + G^{n(\epsilon)}],
\]

where we are denoting the indicator function of the set \( G^i \) by the symbol \( G^i \). Clearly,

\[
\tilde{\phi} \leq \phi \leq \tilde{\phi} + \epsilon.
\]

Thus, it will suffice to prove the theorem for \( \tilde{\phi} \) and, consequently, \( \epsilon^{-1} \tilde{\phi} \). So, without loss of generality, assume from now on that

\[
\phi = G^1 + G^2 + \ldots + G^n
\]

where \( G^1 \supseteq G^2 \supseteq \ldots \supseteq G^n \) are \( G_\delta \) subsets of \( Z^N \).

Define a new game theory problem with state space \( X^* \), action sets \( A^* \) and \( B^* \) and law of motion \( q^* \), where

\[
X^* = X \cup Z \cup Z^2 \cup \ldots,
\]

\[
A^* = A
\]

\[
B^* = B
\]

and

\[
q^*((a, b, x_1)) = q(x_1) \text{ if } x \in X
\]

\[
q^*((p(a, b, x_{n+1})) = q(x_{n+1}) = q(x_n, a, b) \text{ if } p \in Z^n, n \geq 1,
\]

where \((a_n, b_n, x_n)\) is the \( n \)th coordinate of \( p \).

Set \( Z^* = A^* \times B^* \times X^* \) and define \( \psi : Z^N \rightarrow Z^{*N} \) by setting

\[
\psi((x_1, x_2, \ldots)) = ((x_1), (x_1, x_2), \ldots).
\]
By [5, Lemma 6.6], get a set $S^i \subseteq U_{m \geq 1}Z^m$ such that $G^i = \psi^{-1}(\{S^i,i.o.\}), i = 1, 2, \ldots n$. Without loss of generality, assume that $S^1 \supseteq S^2 \supseteq \ldots S^n$. Let $u$ be defined on $X^*$ by the formula

$$u = S^1 + S^2 + \ldots + S^n.$$  

It is now not hard to verify that for every $\bar{z} = (z_1, z_2, \ldots) \in Z^N$,

$$\phi(\bar{z}) = G^1(\bar{z}) + G^2(\bar{z}) + \ldots + G^n(\bar{z}) = \limsup_k (S^1 + S^2 + \ldots + S^n)((z_1, z_2, \ldots z_k)).$$  

So, if we define $u^*$ on $Z^*N$ by the formula

$$u^*((z_1^*, z_2^*, \ldots)) = \limsup_k u(z_k^*),$$

then $\phi = u^* \circ \psi$. Let $u^*$ be the payoff function for the new game.

Fix $x \in X$. We now have two games with initial state $x$: the original game $G_x(\phi) = (X, A, B, q, \phi)$ and the new game $(X^*, A^*, B^*, q^*, u^*)$. By imitating the proof of Lemma 6.1 in [5] and using the fact that $\phi = u^* \circ \phi$, it is easy to see that the two games are equivalent, i.e., the upper values of the two games are identical and the same holds for their lower values. But the main result of [4] states that the new game has a value. Consequently, the game $G_x(\phi)$ has a value. Moreover, since the value of $G_x(\phi)$ coincides with the value of the new game, it can be calculated by the algorithm of [4].

We remark in conclusion that Theorem 5.1 subsumes as special cases all known results about the existence of a value for two-person zero-sum stochastic games with a countable state space, finite action sets and payoff either total discounted reward or total (undiscounted) positive reward or average reward per day. It also yields a value for the recursive matrix games of Orkin [7] with a countable number of matrices, a problem which is posed as open in [7]. However, our Theorem 5.1 does not include the result of Orkin cited in the Introduction.

REFERENCES

