Random Grammars: A New Class of Models for Functional Integration and Transformation in the Biological, Neural, and Social Sciences

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The general problem which arises in investigating the capacities of complex systems to adapt lies in understanding both the functional and the dynamical order which integrates these systems. E. coli "knows" its world. A wealth of molecular signals pass between a bacterium and its environment, which includes other microorganisms, and higher organisms. The signals entering the bacterium are harnessed to its metabolism and internal transformations such that, typically, the cell maintains itself, replicates, and passes is organized processes forward into history. Similarly, a colony of E. coli integrates its behavior. The organisms of a stable ecosystem form a functional whole. The niches occupied by each jointly add up to a meshwork in which all fundamental requirements for joint persistence are met. Similar features are found in an economic system. The sets of goods and services comprising an economy form a linked meshwork of transformations. The economic niches occupied by each allow each to earn a living and jointly add up to a web in which all mutually defined requirements are jointly met. Both biological and technological evolution consist in the invention of profoundly, or slightly novel organisms, goods or services which integrate into the ecological or economic mesh, and thereby transform it. Yet at almost all stages, the web retains a kind of functional coherence. Furthermore, the very structure and connections among the entities sets the stage for the transformation of the web. In an ecosystem or economic system, the very interactions and couplings among the organisms, or goods and services create the conditions and niches into which new organisms, or goods and services can integrate. The "web" governs its own possibilities of transformation.

Similar functional integration of roles, obligations, and institutions apply at societal levels. The revolution occurring in East Europe and the U.S.S.R. in these anni mirabili, are accompanied by a sense that the Soviet system is an integrated whole with the property that if one, or a few features are removed or altered, the entire system must transform to something quite different - and whole. In June 1989 the Communist leaders in China saw fit to kill their
students in Tienamin Square. Why those leaders did so is clear: The students were demonstrating for increased democracy. Their government feared the consequences would transform Chinese Communism. In short, the puzzle is not to understand what China's leaders did, but rather to understand what they knew. In a real and deep sense the Chinese government knew that, were a few features of their system altered, the entire edifice stood in danger of dramatic transformation. What, indeed, did they know?

In the biological and social sciences we badly lack a body of theory, indeed even a means of addressing these issues: What is a functional whole and how does it transform when its components are altered? In this article I shall develop an outline for a fresh approach to these important issues. The approach is based on the use of random grammars. The objects of the theory are strings of symbols which may stand for chemicals, goods and services, or roles in a cultural setting. Symbol strings act on one another, according to the grammar, to yield the same or other symbol strings. Thus, the grammar specifies indirectly the functional connections among the symbol strings. It defines which sets strings, acting on other sets of strings, produce which sets of output strings. These mappings are the functional couplings among chemicals in a protoorganism, among a population of organisms in an ecosystem, and become the production technologies in an economy. Diverse grammars model diverse possible chemistries, or possible production technologies. By studying the robust features of functionally integrated systems which arise for many grammars that should fall into a few broad "grammar regimes", it should be possible to build towards a new theory of integration and transformation in biological and social sciences. Among the features we will find are phase transitions between finite and potentially infinite growth in the diversity of symbol strings in such systems. This phase transition may well underlie the origin of life as a phase transition in sufficiently complex sets of catalytic polymers, and a similar phase transition may underlie "take off" in economic systems once they attain a critical complexity of goods and services which allows the set of new economic niches to explode supracritically.

My suggestion to study random grammars grows from work initiated by myself, in investigating autocata-
lytic sets of polymers, thereafter carried on in collaboration with Doyne Farmer, Richard Bagley, Norman Packard, and Walter Fontana. In particular, I believe the recent extensions by Walter Fontana concerning autocatalytic sets are exciting and important. All point towards a new way to investigate the emergence of functional integration and adaptation in complex systems.
Section I. Infinite Autocatalytic Sets and the Origin of Life as a Phase Transition

The invention of autocatalytic sets lay in an attempt to understand whether the origin of life necessarily required the self complementarity of RNA or DNA molecules, where the plus single strand is a template for its minus strand complement. It is just this feature which has commended DNA or RNA as the "first" living molecules, and has seemed, in principle, to rule out proteins, more readily formed in abiotic conditions, as the first living molecules. But what if a set of proteins might collectively catalyze their own formation from some simple building blocks, such as amino acids? In principle, such autocatalytic sets are nearly inevitable. As we note thereafter, the same principles apply to the emergence of self reproducing sets of single stranded RNA molecules. Here, in outline, is the model:

Consider a set of peptides, where a peptide is a short sequence of amino acids, each a choice of one among several or twenty types. As the maximum length of such sequences, $M$, grows, the total number of polymers grows exponentially, and is nearly $20^M$. This is an old idea. There are five new ideas. First, consider the most primitive reaction among peptides or single stranded RNA polymers. It consists in ligating two polymers into a longer polymer, or cleaving one into two shorter fragments. Such chemical reaction transformations can be represented by a triad of directed line segments leading from the two shorter fragments to a node, and from the node to the larger polymer constructed by uniting the two smaller polymers in a given left to right order. Now construct a reaction graph, showing all polymers as circular nodes, connected by reaction triads whose nodes are square. Use of arrows from the two smaller polymers to the square reaction node and from it to the larger polymer uniquely specify the substrates and product of the ligation reaction. Recall that reactions are reversible. Thus, the same triad specifies the reverse cleavage reaction. Color the reaction line segments black to indicate that the reactions are not catalyzed. This collection of nodes and directed lines represents the uncatalyzed reaction graph.

The second new idea is to consider the ratio of reactions to polymers. Simple calculations show that the ratio of the number of reaction triads to the number of polymers is about $M$. This is a central observation, which as we shall see, holds in very many contexts. The number of components, here peptides or RNA sequences, increases exponentially as $M$ increases, but the number of transformations among the components, here reactions, increases even faster. Thus the ratio of potential transformations among the components to the components increases as $M$ increases.
The third new idea is to note that peptides or RNA polymers themselves can catalyse the ligation and cleavage of peptides or RNA polymers. Therefore, given a model showing which polymers catalyze which reactions, we can ask whether a set of polymers up to length M contains an autocatalytic subset. An autocatalytic subset of polymers is one having the property that each polymer has at least one reaction which forms it catalyzed by some member of the set, and that connected sets of catalyzed transformations lead from some maintained "food set" to all the polymers in the set. As we see next, under a wide variety of models for which polymers catalyze which reactions, autocatalytic sets "crystalize". Note in prospect that since peptides or RNA sequences can be modeled as "strings" of letters, and these strings "operate" on one another, that autocatalytic sets are just a kind of "algebra" or "algorithm" for mapping strings into strings. An autocatalytic set will be some kind of identity under that cluster of operations.

By hypothesis, none of these uncatalyzed transformations occur. Hence a "soup" is unchanging. The simplest model of which polymer catalyzes which reaction is that each polymer has a fixed probability, P, of catalyzing each reaction. This is not a chemically realistic hypothesis, but is useful for the moment. Given this hypothesis, each peptide is asked in turn if it catalyzes each reaction, and answers "yes" with probability P, and "no" with probability 1 - p. Note down, for each reaction, which polymers catalyze it. Color any reaction which is catalyzed red. Thus red triads demark catalyzed reactions. Ask all polymers which reactions, if any each catalyzes. When this process is complete, some fraction of the reaction triads may be colored red. This is the catalyzed reaction subgraph of the uncatalyzed reaction graph.

The fourth new idea is that when the maximum sized polymer, M, is large enough compared to any fixed probability of catalysis, P, then an autocatalytic set will crystalize. The intuition is simple. As M increases, the ratio of reactions to polymers increases. And the number of polymers increases. On average each polymer catalyses a number of reactions equal to P times the number of reactions. As the ratio of reactions to polymers increases, by M increasing, there must come a critical size, Mc, at which almost every polymer has its formation catalyzed by some polymer. This is a phase transition in which a large connected cluster of catalyzed transformations is present in the catalyzed reaction subgraph. Careful analytic results show that, indeed, such a phase transition occurs, related to the giant components of random graphs. After this transition, autocatalytic sets exist. Thus, in a critically complex mixture of catalytic polymers, autocatalytic sets will emerge. On this view, which I hold which fair conviction, life is an emergent "crystalization" of self reproductive metabolisms, based on polymer chemistry, chance, and number, rather
The fifth new idea is that such autocatalytic sets can be "infinite" or finite. Modify the model as follows: Consider a set of polymers up to length L. Call this the "food set". Imagine that all polymers up to L are maintained by exogenous supply. Allow these to undergo ligation and cleavage reactions, if catalyzed by members of the food set, and form polymers of any length ranging up to 2L. In turn, these new polymers are present as potential substrates, products and enzymes, together with all polymers in the food set. Iterate the procedure to assign polymers, at random, those reactions they may catalyze. Update the set of polymers up to length 4L which may now be formed. Over iterations, the set of polymers catalyzed "out of" the food set may continue to increase in diversity without limit. In this case, the growth of the catalyzed reaction graph, and set of polymers formed is unbounded, and the graph growth can be said to be supracritical. The graph of polymers engendered by this process of catalysis is infinite.

Conversely, the growth of the catalyzed reaction graph may stop at some point. No new polymers may be added to the system. Then the growth of the graph is subcritical.

Fairly simple algebraic arguments suffice to define a phase transition in a parameter space whose axes are the probability of catalysis, P, and the size of the food set, a function of L. A line partitions two regions, in the subcritical region, either the food set is too small, or the probability of catalysis is too low, and graph growth is finite. Above the transition, graph growth is infinite. The critical scaling law is

$$B^{Lc^*} = (1/2P)^{1/2} \text{ or } P = B^{-2Lc^*}$$

where B is the number of kinds of amino acids considered. And since the same story obtains for single stranded RNA molecules, B is the number of kinds of nucleotides considered.

It is easy to unite the two pictures given and show that, in principle, supracritical, hence infinite, autocatalytic sets can exist.

It is important to stress that in this analysis I have eschewed any discussion of the thermodynamic requirements driving synthesis of larger polymers from smaller ones. I shall continue to do so for the moment. That is,
we are considering only the formally allowed transformations among hypothetical polymers engendered by catalysis of reactions, not yet the issue of the actual flow of real polymers over the space of catalyzed reactions under specific physical conditions.

\textit{RNA World, Chemically Plausible "Template Match Rules" Yield Finite and Infinite Autocatalytic Sets}

The existence of autocatalytic sets persists with more plausible models, based on template complementarity in single stranded RNA sequences, of which polymers catalyze which reactions. The work that follows has been carried out with Rick Bagley at SFI. Consider a population of single stranded RNA molecules. The hypothesis that life started with such RNA molecules is widespread, both because RNA molecules, called ribozymes, have now been found to catalyze reactions, and because template replication of arbitrary single stranded RNA molecules seems so reasonable. Each strand is to line up free complementary nucleotides, then catalyze the joining of these to form a complementary strand which will melt off and repeat the cycle. Note that, were this to work, then any single stranded RNA molecule would be able to lead to synthesis of its \textit{precise} complement, and via the latter, to its own synthesis.

Such a +/- pair would, of course, be an autocatalytic set. The + strand catalyzes the formation of the - strand, and vica versa.

In practice, unfortunately, this autocatalytic template replication of an arbitrary single stranded sequence has not yet succeeded. The only case which works uses a + strand comprised by C and G nucleotides with C > G in composition. But the resulting - strand has G > C and fails to work as a template.

Suppose such reactions were, in general, to work, and note a further feature. Here we have a \textit{finite} autocatalytic set, a + and - pair of template complement strands replicating themselves, but not necessarily reaching out into sequence space and replicating or catalyzing the formation of other sequences. Thus, we reach a simple, important conclusion. \textit{In principle, autocatalytic sets can be either finite or infinite.}

Indeed, \textit{finite RNA autocatalytic sets exist in practice. The single stranded RNA hexamer GGCGCC acts on two trimers, CCG and CGG, binds them via complementary base pairing, ligation, and releases the resulting hexamer which, upon examination, is identical in 3'-5' order. Thus, a "soup" supplied with the hexamer and a popu-}
lation of trimers will form a finite autocatalytic set with three sequences as members, and convert the trimers largely to hexamers.

Note the critical point that the hexamer is acting as a specific ligase, recognizing specific trimers, not as a general polymerase. Thus, actual chemistry supports specific ligation by single stranded RNA molecules.

Strikingly, single stranded RNA molecules and specific ligation based on template recognition supports not only finite autocatalytic sets, but in principle, supports supracritical, infinite autocatalytic sets. Bagley and I have modified the rules for catalysis. We require that the potential catalytic strand "template match" the left and right three or four nucleotides on the two potential substrate molecules, either perfectly or with perhaps one mismatch allowed. Matching here is base pair complementarity. Only if such sequences match does the prospective "ribozyme" have a chance to be an actual catalyst for the reaction. The chance is proportional to the "matchstrength", and polymer length. Again, leaving out thermodynamic issues, we find that with a critical complexity of polymers in the "soup" we obtain apparently infinite autocatalytic sets. The criteria for this judgement is that the number of new polymers being added to the system grows supraexponentially over updates of the catalyzed reaction graph.

Thus, the RNA world, with string matching, can give rise to both finite and infinite autocatalytic sets.

Evolution and Coevolution in Autocatalytic Sets.

At present Rick Bagley and Walter Fontana are implementing code to follow the capacity of autocatalytic sets to evolve. A "shadow" set of polymers derived by spontaneous reactions among the polymers in the set is assumed to form at low frequency. The "shadow set" is a suggestion of Doyne Farmer. If among the shadow set, one or a cluster of new polymers can catalyze their formation from themselves and the existing autocatalytic set, they are added to the set. Thus, the set evolves to a new autocatalytic set. In turn, old polymers may die out of the set, due to addition of real thermodynamic criteria which I will not discuss here. Thus, evolution can occur in a space of polymers and transformations among the polymers. In addition, autocatalytic sets are favorable objects to study coevolution. We need merely define interactions between such sets, whereby sequences made in one set can migrate to and enter another set. Such signal sequences may integrate into, or disrupt, the functional organization of each set in ways
I will return to below. But, in short, such couplings and their coevolution affords the opportunity to study the emergence of shared functional integration in coevolving reproducing entities.

*Emergence of a Connected Metabolism*

Similar arguments, here briefly presented, apply to the emergence of a connected metabolism. Consider all organic molecules, counted in terms of the number of carbon atoms per molecule. As the number of atoms per molecule goes up, the number of organic molecules explodes very rapidly. But the number of legitimate reactions by which they transform rises even faster. Indeed, consider, in general, reactions with two substrates and two products. It might be thought of as a mapping of pairs of organic molecules into pairs of organic molecules. Occupied "cells" in such a matrix show the legitimate chemical reactions. It is easy to see that if each pair were able, on average, to undergo only one reaction, then the number of reactions is crudely the square of the number of organic molecules. Now consider two related hypotheses. First, cast a mixture of RNA or peptide sequences upon this space of organic reactions. If each sequence has some chance of catalyzing any reaction, then if enough sequences are cast upon the space, or if the space is complex enough, connected sequences of reactions will crystalize as a giant component in the reaction graph. A connected metabolism will emerge whole. The second related hypothesis notes that organic molecules can catalyze organic reactions. Under a variety of hypotheses about the distribution of catalytic activity among the organic molecules, autocatalytic sets should emerge.

*Functional Integration*

Autocatalytic sets exhibit an emergent *functional integration*. Once the set of chemical reactions, ligation and cleavage, and the set of polymer strings, and the assignment of which strings catalyze which reactions is made, the rest is emergent. The notion of function is contained in the idea of which strings catalyze the transformation of which other strings. The idea of an autocatalytic set is precisely that of *catalytic closure*. This is a first, root, central image of a *functional whole!* The members of an autocatalytic set get themselves "made" by members of the set. All niches necessary are filled. It is just this sense of catalytic closure that is the start of an image of functional wholeness that I want to pursue. However, as we shall see in considering "catalytic jets", the requirement of closure is too strong.
Autocatalysis Must Be Very General

These ideas are very general. They rest upon the ideas that there is a set of objects, a set of transformations among the objects, and that the objects can themselves mediate the transformations. One then notes that for a wide variety of such objects and transformations, the number of transformations grows more rapidly than the number of objects, hence with a wide variety of distributions of which objects mediates which transformation, autocatalytic sets should emerge. One expects the ideas to have wide applicability.

Fontana's AlChemy.

Walter Fontana has recently generalized the analysis of autocatalytic sets and catalyzed transformations in a useful way. Note that, in the model of the origin of life, one is describing mappings of strings into strings, mediated by strings. Thus, as remarked above, this is in general, some kind of algebraic or algorithmic transformation in which autocatalytic sets are certain kinds of identity operations of a cluster of the transformations and objects.

Fontana borrows a powerful algorithmic language derived from the lambda-calculus, invented to be as powerful as universal Turing machines, and the progenitor of Lisp. Fontana's idea explicitly, is to exploit the general idea of strings acting on strings as algorithms. In this hypothetical chemistry, Fontana does not require that mass be conserved. Two strings collide, the first is the "program" which acts on the second as an input. By construction, most strings are legal both as program and as input. Thus, most collisions between strings transform the second string into some single new string. Fontana defines a "Turing gas" in which a random collection of strings is placed in a "chemostat". After each productive collision between strings, the number of strings has increased by one. To supply a selective condition, Fontana randomly removes one string, hence holds the number of strings in the chemostat constant.

Fontana has carried out three kinds of numerical experiments. In the first, a set of 700 strings is allowed to interact by random collisions. At first only new strings are generated. But over time, more and more strings which are generated already have identical copies in the chemostat. Eventually a closed set of strings, an autocatalytic set,
emerges. In this first set of experiments, the autocatalytic set is dominated by a general replicase, or a sequence that can copy itself and any other sequence. It is equivalent to a ribozyme which might copy itself and all others.

In the second set of experiments, Fontana disallows copying strings. Nevertheless, closed collectively autocatalytic sets of strings emerge. Thus one set had some 45 kinds of strings present which mutually transformed into one another. These sets are the direct analogues of the autocatalytic polymer sets discussed above.

In the third numerical experiment, Fontana injected sets of 20 random strings into an evolving chemostat. He found that the terminal autocatalytic set differed from that which would have occurred without exogenous perturbation.

Section 2. Jets and Autocatalytic Sets: Towards a New String Theory

Whether we are considering the transformation of chemicals, or goods and services in an economy, or a variety of other cases, it seems useful to consider the infinite set of binary strings as the "objects" under analysis. Then, in general, strings or sets of strings act on strings or sets of strings to yield strings. In general, such transformations are just mappings, or algorithms. The set of strings operated upon can be one or many. The set of strings carrying out the operations can be one or many. In general, the set of transformations will increase more rapidly than the set of strings. Thus, the general question is this:

For various kinds of random or nonrandom mappings of strings into strings, what kinds of "sets" of strings emerge?

What we need, in general, is a way of generating families of algorithms, or grammars, or finite state automata which realize those algorithms, or grammars, and discover the kinds of functionally "generative" sets we obtain. I turn next to some intuitions about those sets, then return in the next section to consider ways of studying the space of possible grammars.

String Set "Geometries": Jets, Lightning Balls, Mushrooms, Eggs, Fixed, Traveling, Wobbly Ergodic and Hairy Eggs, Filligreed Fogs, and Pea Soups.
Consider first a "Jet". Imagine a rule for polymer sets in which any string only catalyzed the ligation of strings both of which were larger than the catalyst string. Then, by construction, no feedback loops could form. All catalyzed transformations would lead to ever larger strings.

Let me define a Jet as a set of transformations among strings from some maintained "founder" set of strings, analogous to the food set, \( S_0 \), having the property that under the algorithmic transformations among the strings, each string is only produced by a unique set of "parent" strings, and is produced in a unique way. This is probably sufficient, but more than necessary. In any case, it leads to a Jet of string productions which never cycles back on itself.

Note that a Jet might be finite, or might be infinite.

A Lightning Ball is a Jet cut free from its founder set, free to propagate through string space until it dies out, if the jet were finite, or propagate forever, if the jet were infinite. Presumably a periodic or a quasiperiodic Lightning Ball which orbits back to the starting set of strings, or near the starting set of strings, is possible. The "orbits", defined as the succession of sets of strings in the Lightning Ball, in string space might be chaotic or ergodic.

Let me define a Mushroom. The first example is an autocatalytic set of polymers growing forth from a maintained "food set". Here a set of transformations jets up via a kind of stem free of feedback loops, then feedback loops begin to form creating the head of the Mushroom. In effect, a Mushroom is a Jet from a maintained founder set, with feedback loops.

Mushrooms are models of functional "bootstrapping". An immediate example is an autocatalytic peptide set with a sustained metabolism of coupled transformations from the food set. But perhaps another example is the technological evolution of machine tools. For example, the first tools were crude stones, then shaped stones which enabled formation of better tools, used to dig ore, then metal tools, then ultimately, the development of machine tools which themselves generate tools such as axes, chisels, and machine parts for other machine tools. Presumably the onset of agriculture is a similar example. Many more must exist in economic and cultural evolution, as well as organic evolution.
Like Jets, Mushrooms can be finite or infinite.

Next consider the Egg. The examples of hexamer and trimer RNA sequences which reproduce only themselves in RNA sequence space, and Fontana’s two autocatalytic sets are eggs, whole in and of themselves. Eggs are self sufficient sets of algorithmic transformations with no need of a "stem" from a maintained founder set. In an egg, strings can produce arbitrary strings, hence can enter a closed set which finds only itself, free of all other strings. Let me reserve the term Egg for finite closed autocatalytic sets. An unchanging finite egg is a kind of identity operator in the process algebra or grammar by which strings act on strings such that the collection of processes produces precisely and only itself. Eggs may prove to be useful models of self confirming mythic or even scientific conceptual systems by which the outside world is parsed. They may also prove useful as models of cultural identity, integration and wholeness.

Presumably Eggs come in several types. We have already considered the Fixed Egg which maps into itself. But eggs might move through string space, creating Traveling Eggs. The former correspond to autocatalytic sets which are closed and hold to a fixed set of sequences and transformations. Traveling eggs, rather like Lightning Balls, might contain feedback loops, but change composition in sequence space in various ways. Presumably, Wobbly Eggs, which orbit among a periodic set of sequences, or quasiperiodic set, might be possible; so might Chaotic eggs and Ergodic eggs exist. The set of strings in an ergodic egg would wander randomly over string space as the egg traveled. In addition, Hairy Eggs would be finite objects from which Jets or Mushrooms may extrude, perhaps stochastically if the production rules are activated probabilistically. Fontana may have found such structures with a stable core metabolism sending out a fluctuating flare of other strings.

The Filigreed Fog is an infinite supracritical autocatalytic set which may or may not have a stem to a sustained founder set, or, like an egg, might also float free. Unlike an egg, however, it is not bounded. Nevertheless, the Filigreed Fog is limited in that there are at least some strings which can never be generated by the set.

Finally, there is the Pea Soup, defined as an infinite set which, in principle, will eventually include all possible strings. It is intuitively clear that the autocatalytic set generated by the model in which each polymer has a fixed chance of catalyzing each reaction will form a Pea Soup if it is supracritical. Ultimately all strings could have
their formation catalyzed by some string.

**Evolution and Stability of Functional Sets.**

Among the obvious questions about such sets are their stability and capacity to evolve. Consider an Egg. How many Eggs does the algorithmic set contain? A few? Many? Given a definition of 1-mutant variants, is in Egg stable to all 1 mutant variations in its composite strings? Two mutant variants? Etc. Thinking of Eggs as attractors, how many are accessible from any Egg, by how much of a mutation in the set of strings present? For example, Fontana began to study this by injection of exogenous strings. Can one jolt an Egg to another Egg? Similar questions apply to all the kinds of structures depicted. Such questions bear on the stability and capacity to evolve by "noise" of such functional sets.

Note that transformation of a Jet to a Jet, or Mushroom to another Mushroom, or Egg to Egg, or between such types of sets by a perturbation or mutation begins to get at our intuitions that the Soviet or Chinese political system is fragile, that a few minor changes in the coherent structure must lead to the replacement of many or most functional linkages.

**Decidability Problems**

A number of issues may be undecidable. For example, whether a given set of founder strings in a given algorithmic set of rules is subcritical or supracritical might be such an issue. It appears intuitively related to the halting problem. Similarly, in a Filligreed Fog, it may not be formally decidable that a given string cannot be produced from the initial set of strings by the grammar. I suggest below in an economic context where strings are goods, that such formal undecidability may map into the logical requirement for bounded rationality in economic agents, and an equal logical requirement for incomplete markets. Thus, such models may invite modification of neoclassical economic theory.

**Size Distribution of Avalanches of Change**
In autocatalytic polymer sets, addition of a new polymer may trigger the formation of many new strings, and elimination of old strings. In a technological web, addition of the automobile drove out the horse and many horse trappings. When Fontana injected random strings, a peripheral component of the autocatalytic metabolism tended to change. What is the size distribution of such avalanches? For example, Bak and colleagues have drawn attention to Self-Organized Criticality in their Sand Pile model. At the critical state, a power law distribution is found with many small and few large avalanches. A similar distribution is found in Boolean networks at the edge of chaos, and in certain model ecosystems which have optimized joint fitness. Thus, we are led to ask what such avalanches of "damage" or changes, look like in the various objects discussed. For example, it might be the case for finite Jets that avalanches early in the Jet are large, and late in the Jet are small. Or avalanches might show a common distribution regardless of when unleashed in the lifetime of a Jet. Similar questions arise with respect to Fixed Eggs and Traveling Eggs. Perhaps a power law distribution obtains whenever sets go supracritical. Perhaps that is just when all damage is also infinite.

Note that these questions may allow us to begin to address such issues as the sensitivity of history to small perturbations. For history, too, is an unfolding of transformations among some indefinite, or infinite set of possibilities. Similarly, the evolution of autocatalytic sets in a world of polymers, with coevolution among the sets, captures both historical accidents and a kind of entropic exploration of the world of the possible.

Sets of Strings Acting On sets of Strings: Aggregated Transformations as Machines Tunes the Ratio of Transformations to Strings.

Consider Fontana’s Turing-gas. A thousand strings interact with one another by random collisions. This parallels the studies on autocatalytic sets. Suppose that 100 different types of strings are present. Then the chance that any specific string will undergo an ordered set of 5 of the transformations mediated by these strings is low. Consider instead, a machine, or complex made of a sequential aggregate of 5 kinds of strings, such that any string which encounters the aggregate undergoes sequentially all five transformations. Thus the aggregate is a kind of "machine" made of simple transformations, which ensures a complex set of transformations. Now consider that if there were 100 kinds of strings in the gas, then there are $100^5 = 10^{10}$ of these combined fivefold transformations.
One implication of the use of an aggregate, or ordered set of 5 kinds of strings as a "machine" which acts on a single string or a set of strings, is that the number of "machines" is very much larger than the set of single strings. Since each machine carries out a compound transformation on an input string, this is equivalent to saying that construction of complex machines increases the ratio of potential transformations mediated by one machine to strings. Hence achieving phase transitions to more complex supracritical sets becomes easier.

Another implication is that coordination of 5 strings in an ordered way into a machine alters effective time scales. If strings acted on one another in random pairwise interactions, a vast set of strings would have to be present in the system to assure that all of these complex transformations were sampled in reasonable time. Thus, we can think of the aggregation of primitive strings into aggregates, the invention of machines, as means to mediate specific compound transformations at high frequencies. Clearly this will alter the functional sets formed. This example makes it clear that time scales matter. A set of strings interact by some dynamics, as in Fontana's random collision dynamics. Altering the probabilities of string interactions profoundly alters which sets of composite transformations occur and which sets of strings arise.

The image is not a poor one. The machines in our economy form specific complex objects among a set of many other possible ones.

*From One Chemostat to Many: Coevolution and Phase Transitions.*

By introducing a multiplicity of chemostats which operate on strings internally, and may also exchange strings between the chemostats, we can explore models of coevolution, the emergence of competition or mutualism in biology, or economic trade between economic agents or units. In addition, phase transitions among the kinds of sets generated, jets, mushrooms, and so forth, can take place as a function of the number of chemostats which come to interact. As that number increases, the joint complexity of strings being operated upon can pass critical thresholds. Such transitions may model "take off" in an economy or even intellectual community. I now discuss this in more detail.
The autocatalytic model, and Fontana's model, so far take place in a stirred reactor. All strings can interact with all strings. Consider instead a set of chemostats or boxes. Each box, to be concrete, begins with a sustained founder set, which are its sustainable natural resources. Each set of strings proliferates purely internally. So far this is nothing but the stirred reactor within one chemostat. Now let some of the strings be made for "export only". These exported strings may pass to other boxes. Those other boxes may be identified by spatial location, or some strings may "bind" to the box surface and serve as address strings. In either way, the invention of multiple chemostats or boxes serves to identify "individual" regions of "local processes" which may then coevolve with other such regions.

Among the first questions to consider are these:

Imagine that each box, granted its sustained founder set, yields only a finite Jet by itself. It may be the case when strings can be exported between boxes that some or all of them are lifted to a more complex level of activity. For example, the collective system might form an infinite Jet, a finite or infinite Mushroom, a Filigreed Fog, or even a Pea Soup! The clear point to stress is that collaborative interaction may transform a system from one to another of the types of functional sets. In particular, there may be a critical level of complexity for any given set of algorithmic transformations, leading with high expectation to each of these transitions. If so, what are these thresholds like? Might they, for example, bear on economies which are unable to expand in diversity of goods versus those which can explode? Does this concept bear on the consequences to isolated cultural systems when brought in contact with other isolated systems or a larger world culture? Do they bear on the scientific explosion following the Renaissance?

Consider the question of functional integration between the boxes. Each box can be thought of as a kind of country with natural resources, or a firm interacting with other firms, or perhaps an integrated functional organism. String inputs from other countries or organisms may perturb the internal dynamics of each box. In response, the box may "die", that is the Jet or other process might collapse to a sustained founder set or to nothing, the null set, or it might transform to some other more or less constant functional set. In the latter case, we have an image of entities which alter their internal structure in response to external couplings such that each is internally a stable sustained flux of collaborative processes in conjunction with the couplings to the other boxes. It is an image of stable signal relations among bacterial cells, or perhaps, as we see in more detail below, trade relations among nations endowed with different natural resources and histories of technological development. Are there many alternative attractors to such a
system given the same founder sets to each box? How history dependent is it? How stable to perturbations?

Such coevolving "boxes" literally come to "know one another" and know their worlds. We must consider when and whether such systems are competitive or such systems coevolve mutualisms which optimize mutual growth rate or equivalently, utility. Indeed, I suspect that these processes must occur in biologic and economic evolution.

**Dynamical Stability as Well as Compositional Stability**

The sets we considered above, jets, mushrooms, eggs, fogs, and so forth, deal with the string composition generated by different grammatical or algorithmic rules by which strings interact. But in addition to the composition of such generated sets, it is also important to consider the dynamical aspects of such systems in terms of the "concentrations" of strings of each type over time. For example, an Egg might reproduce itself compositionally at a dynamical steady state or along a limit cycle in string space by which its constituent strings were successively produced. Presumably other orbits might suffice for an Egg to persist. Similar questions arise for jets, fogs, and other potential objects.

A critical difference between string systems and familiar dynamical systems is that the former operate in an indefinitely large state space, the latter in a fixed state space. For example, Boolean networks and other dynamical systems exhibit dynamical attractors in a fixed state space. The functional sets we are considering are, in a sense, evolving in an open state space of strings.

In particular, Boolean networks exhibit three main regimes of dynamical behavior: chaotic, ordered, and complex. The transition between these is governed by a phase transition. In the ordered phase, percolating frozen components where binary variables are fixed in active or inactive states, span the system leaving behind isolated islands of unfrozen elements free to fluctuate from 0 to 1 to 0 in complex patterns. In the chaotic regime, the unfrozen "spins" form a percolating component. In the complex region, which lies at the boundary between order and chaos, the frozen component is just melting and the unfrozen component is just percolating. Avalanches of "damage" due to perturbing the activity of single sites propagate on all length scales in a power law distribution in the complex regime. Damage only propagates a finite distance in the frozen ordered regime, typically within one unfrozen island.
Damage propagates to a finite fraction of all spins in the infinite spin limit in the chaotic regime, exhibiting sensitivity to initial conditions in the chaotic regime.

A variety of tentative arguments suggest that systems in the complex regime on the edge of chaos can carry out the most complex computations and can "adapt" most readily.

Our questions with respect to functional sets concern whether or not the analogues of frozen components, and whether ordered, chaotic and complex behavior. I return below to show why I am quite certain the answer is yes. One important implication is that the dynamical behavior of a set of strings can control how it explore string space. For example, an infinite fog may not be populated because the system cannot pass bottlenecks in string space for dynamical reasons.

Section III. Infinite Boolean Networks and Random Grammars: Approaches to Studying Families of Mappings of Strings into Strings.

In order to study jets, eggs, fogs and functional interactions, we require mathematical models of the algorithmic interactions by which strings act on strings to produce strings. The autocatalytic polymer set with fixed probability, P, of catalysis, the RNA string match rule Bagley and I have investigated, and Fontana's Alchemy are three choices of rules by which strings act on one another. The aim of this section is to consider alternative approaches to generate in some ordered way the set of "all possible" mappings of strings into strings. In fact, this cannot be done in an ordered way. The set of all such mappings involves the infinite power set of binary strings of infinite length acting on itself to produce the infinite power set of binary strings. This class of objects is not denumerably infinite. It maps to the reals. Consequently, any ordered approach to this problem requires simplifying at least to a denumerably infinite set of objects, categorized in terms of some parameters such that mappings of increasing complexity can be studied and such that these mappings fall into useful classes.

The aims of this endeavor should be stated clearly. I believe such mappings, grammars, or algorithms, sending strings or sets of strings, operated upon by strings or sets of strings, into strings or sets of strings, may provide useful models of molecular interactions or molecular machines in organisms, models of production technologies
in economic systems, models of conceptual linkages in scientific ideational, or cultural systems. We surely do not, at this stage, have detailed understanding of such functional couplings among metabolites in organisms, of technological possibilities governing linked production technologies in economies, or among mythic or other elements in cultural systems. The hope is this: By exploring large tracts of "Grammar Space" we may find rather few "regimes" in each of which the same general behavior occurs in the sets of strings generated by the specific grammar. That is, just as exploration of random Boolean networks has revealed three broad regimes, ordered, complex, and chaotic, so too may exploration of grammar space. We can then hope to map these broad regimes onto biological, economic, or cultural systems. Thereby we may obtain models of functional couplings among biochemical, technological, or ideational elements without first requiring detailed understanding of the "physics" or true "laws" governing the couplings of functional elements of those diverse systems. We may find, in short, the proper Universality classes.

I next discuss two approaches to this task. The first explores the representation of mapping of strings into strings via infinite Boolean networks. The second considers the use of random grammars with definable parameters which allow grammar space to be explored.

A Natural Infinite Dimensional State Space Representation of the Mapping of Strings into Strings. Infinite Boolean Networks.

One natural representation for strings mapping into strings is an infinite dimensional state space of discrete objects, for example symbol strings which can be finite or infinite in length. Consider binary strings of length N, where N can increase up to infinity. Order these in counting to infinity, beginning with the two "monomers" 0 and 1, then the four "dimers" 00,01,10,11, followed by the eight trimers, and so on. At each string length, L, there are $2^L$ types of strings. This infinite list of string types whose lengths also increase to infinity can be ordered from a starting point, the monomer 0. Create two infinite matrices. The first "input matrix" is ordered such that each column denotes one specific binary symbol sequence, and the columns begin with the monomers at the right-most column of the matrix, the dimers to the left, the trimers to the left of the dimers, and so forth stretching to infinity in the left direction. Thus each possible symbol strings, from short to long, is assigned one column in the input matrix. The second "response" matrix is simply the mirror image of the input matrix. The response matrix lists the monomers in the leftmost two columns, the four dimers to their right, etc. stretching to infinity in the rightward direction. The
"input" matrix has as its rows all possible combinations of the presence of absence of the possible types of symbol strings, starting with the row (.....0000) on top. There are an infinite number of rows in the input matrix. The cardinality of both the column and the row infinities is, of course, the same.

The positions of "1" values in each row of the input matrix represent which strings are present in that state of the world. The response matrix will show the next state of the world as strings act on strings to produce strings. By construction, the input and response matrices are mirror symmetric, hence to read the "next state" that is formed from each input state, the reading must be flipped from right to left for the input matrix to left to right for the response matrix.

Alternative mappings from the input to the response matrix represent alternative mappings of the set of strings into itself. In order to proceed further, some further definitions are required. Let a machine, M*, be an ordered collection of M strings. Let an input bundle, I*, be an ordered set of I strings. The action of M* operating on I* will yield an ordered output set of strings, O*.

Any row of the input matrix has a finite number of "cells" with the value 1, representing the fact that each row represents a unique combination presence or absence in the "world" of strings up to some length. The possible machines built of these strings might be limited to a specific maximum number of strings, for example, 5, or might range up to the finite total number of strings present in that state of the world. Call the latter maximum size machine "unbounded" in the sense that as rows further down the infinite input matrix are considered which represent the presence of yet more symbol sequences, still more complex machines can be built. Due to the ordered way that the input matrix is constructed, it is possible given any constraint on which ordered sets of strings count as legitimate machines, M*, or input bundles, I*, to uniquely number each machine, M*, and input bundle I*. In a moment I shall use such unique numberings to produce a deterministic mapping from the current state of the "world" to the next state.

The choice to include all possible "unbounded" machines as legitimate machines or unbounded input bundles as legitimate bundles, specifies the the power set of strings operating on itself as the mathematical entity of interest. This is clearly the widest interpretation. It allows the generation of the maximal number of strings in the pres-
ence of a fixed set of strings that is possible under any interpretation of the kinetics in which strings collide with one another and act on one another. Other choices are more limited. For the moment I therefore choose the widest, full power set interpretation of machines and input sets.

If a string is acted upon and transformed, we need to choose whether the initial string remains in the system or not. The natural interpretation is that the string is "used up" or disappears in the transformation. (Note in chemistry, the back transformation always occurs. This is not in general the case, however.)

With these assumptions, the next state of the world is a mapping from the present state, given by some Boolean functions in the response column. Such a system can be thought of as a discrete time, autonomous synchronous infinite automaton. The dynamics of this infinite automaton gives the way strings engender strings in the potentially infinite space of strings.

Four further assumptions lead us to a canonical and ordered way to generate a sensible series of families of transformations.

1) Note that we can parameterize such transformations by the largest size machines allowed, and the largest size input sets to a machine allowed, $M^*$ Thus, the system might at the moment have 1000 strings, but only machines with 5 or fewer strings, or input bundles, $B^*$ with 5 or fewer strings might be allowed. In any row in the input matrix there are a finite number of sites with "1" values, corresponding to strings present in the system, say $N^*$. The the maximum number of machines is $(N^*)^5$. The maximum number of input bundles is similarly $(N^*)^5$. The product of these is the maximum number of pairs of "machines" and "input bundles". Determinism demands that for each, there is a unique outcome.

2) We need a rule which limits the number of output strings, given the number of input strings as well. Call this a limit on the "output spray", $S^*$. Given that as well, then given $N^*$ in the input row we have a maximum limit on the number of strings present in the next state of the world. Call this $N'$.

3). Next, note that we can choose to delimit the maximum length of a new string produced by machines whose maximum member length is $M$, and whose maximum input bundle string length is $L$, to some finite bound which in-
creases with $M, L$, or $M+L$. Thus, in the origin of life model from a food set, at each iteration, the maximum string length doubles because one imagines ligating two strings present in the system. Any such bounding choice is a fourth parameter, which in effect creates an expanding cone down the rows of the response matrix. The cone asserts that maximum string length can only grow so fast as a function of lengths of strings already present in the system.

Given these bounds, then we have, for each row of the input matrix, a bound on the maximum number of strings which can be present in the next state of the world, and a bound on the maximum length of those strings. (Note, parenthetically, that a constraint exists between the maximum rate of cone expansion of lengths of strings, and the total number of output strings from any input state of the world. There must be enough possible strings in the "space" allotted, to accept the new strings).

The Quenched Deterministic Version: Mapping the Infinite Powerset of Binary Strings into Itself.

The mathematical object we are considering in the Boolean idealization where all allowed transformations occurs is really a mapping of the infinite power set of $N$ strings into itself. That is, consider a row of the input matrix. It contains a set of $N$ strings. But the power set of ordered pairs of strings, ordered triads of strings,...ordered $N$-tads of strings, is just the set of all possible machines, $M^*$, constructable from those strings. Similarly the sets of single strings, ordered pairs of strings, etc are the set $B^*$ of possible input bundles. $B^*$ and $M^*$ are the same power set in the limit when machines and bundles with $N$ strings in them are allowed. As remarked above, identify each unique ordered set of strings which is a machine, $M_i$, with a unique number. Similarly, identify each unique ordered set of strings which is an input bundle, $B_i$ with a unique number. Then the pair of numbers, $i,j$, specify a unique machine input pair, hence must always have a fixed output bundle, $O_{ij}$. The output, of course, is bounded by the output spray $S$.

Given this, then it is possible to define for each machine input pair, regardless of which state of the world it occurs in, hence which row of the input matrix, a unique output bundle. This assures both determinism, and constraints within the family $M, B, S$. Since the number of strings and their lengths comprising machines, input bundles, and output sets are all finite and bounded for any unique machine input pair, it follows that we can generate all possible finite number and length legitimate output sets which might be generated by the unique machine and in-
Thus, it follows that, in terms of the parameters giving maximum machine, bundle and "output spray" sizes, and cone angle or string length amplification factor, as well as the deterministic machine input pairs mapping into unique output bundles, we can consider all possible mappings of the infinite set of strings into itself. Hence within these parameterizations we can explore all possible dynamical behaviors this family of systems.

I note that this construction is also effectively simuleable. It is feasible to use the identifying numbers of each machine and input bundle, i,j, as seeds to a random number generator which specifies uniquely the output bundle, O_{1,j}. Thus, it is not necessary to hold massive memory files for the mapping between each input state and its successor output state of the world. Such simulation does not seek to study all possible mappings, but a large random sample of mappings for different values of the parameters of the model.

*The Annealed Model*

Given the bounds on the response matrix above, we may consider a simpler annealed model which may prove useful. Consider, for each row of the input matrix and the bounds on amplification, output spray and so forth, all possible ways of filling rows in the response matrix with 1 and 0 values, consistent with those constraints. Each way corresponds to a well defined transformation of the infinite set of strings into the infinite set of strings, and allows an expanding cone of complexity. However, this model does not preserve deterministic dynamics. It is an annealed approximation to a deterministic "grammar" or mapping of the infinite set of strings into itself. The lack of determinism is easy to see. Consider two rows of the input matrix in which strings S1...S5 are present, but in the second of these rows, string S6 is also present. By determinism, in the second row, all the machines, input bundles and transformations which might occur in the first row are also present, hence must occur in the corresponding next state of the world. But under the model above, filling the response matrix rows in all possible ways, such determinism is not guaranteed. Instead this model is an annealed approximation to a deterministic dynamics, whose statistical features may prove useful to analyze, as has proved the case with Boolean networks.
The concepts of Jets, Fire Balls, Eggs, Traveling and Fixed Eggs, Filligreed Fogs and Pea Soups, etc. are all clear in either the deterministic or annealed picture as trajectories from a maintained source set (Jets Mushrooms, etc.) or "free" dynamics (Eggs, Fire Balls, Filligreed Fogs, Pea Soups, etc.)

Since these systems are just infinite Boolean networks, the concepts of dynamical attractors, ordered, chaotic, and complex behaviors, carry over directly. But in addition, in comparison to a fixed state space, we have here the idea of attractors in composition space, namely the sets of strings which comprise the system as well as the dynamical behaviors evidenced among the set of strings, be they eggs, mushrooms or filligreed fogs.

One can begin to guess at the relation between dynamics and composition space. The Boolean idealization shows the set of all possible transformations from the current set of strings into the next set of strings. In any other dynamics, for example a Turing gas model of collisions, not all machines and input bundle pairs will occur at each moment will occur, hence only a subset of all transitions will occur. In particular, one might want to model the presence of an "inhibitor" string which, when present, unites with a machine and reliably blocks its action, just as repressor molecules bind to cis acting sites and block transcription. Once one allows inhibition of transitions in this way, then the dynamics which occurs can be chaotic, ordered, or complex.

Consider the case in which, on the infinite graph, the graph growth process creates an leaky egg which emits a narrow infinite jet. Will the infinite jet actually occur? The dynamics itself will control the subset of the composition set explored. For example, the dynamics of the system might cut off all transformations at the base of the infinite jet, so no strings will actually be formed which flow up and create the jet. Clearly, it will be easier to control this process if the dynamics of the system were in the ordered regime, rather than chaotic. Then all string processes from the egg to the jet entrance might be inhibited. Under chaotic dynamics in the egg, firing of strings at the base of the jet would be hard to prevent. Thus, control of which subset of the composition set actually occurs is clearly more readily done if there are ordered dynamics than chaotic dynamics. But conversely, achievement of ordered dynamics in Boolean networks requires control over the number of inputs per variable, and over the biases in the Boolean functions. Both controls will be easier to maintain in a finite egg, than in an infinite mushroom, filligreed fox, or pea soup. In these cases, the elaboration of feedback connections to each string is roughly unbounded. Thus, these systems are more likely to exhibit chaotic dynamics, and thus to explore fuller reaches of their possible composition set,
than will finite eggs with orderly dynamics.

Obviously, finiteness in real physical chemical systems is also controlled by thermodynamics, in economies by the costs of raw materials, etc. However, in worlds of ideas, myths, scientific creations, cultural transformations, etc., no such bound may occur. Thus it is of interest to see how such algorithmic string systems can control their own exploration of their possible composition set by dynamic control over their actual processes.

The generalization to the case with "boxes" is obvious. It is equivalent to a set of "linked" Boolean nets, ie those which share some "external" variables.

Random Grammars

While the use of infinite Boolean systems should prove useful, the use of random grammars may be even more feasible and useful. Grammars range from simple regular languages to context insensitive and context sensitive to recursively enumerable. These most powerful grammars are known to be as powerful as universal Turing machines. Any grammar can be specified by a list of pairs of symbol strings with the interpretation that each instance of the "right" member of the pair in some "input string" is to be substituted by the corresponding "left" member of the pair. Thus, were the sequences (110011) and (0011) such a pair, then starting with a given input string, any instance of (0011) would be replaced by (110011). Effectively carrying out such a transformation on an initial string requires a precedence order among the pairs of symbol sequences in the grammar, and a means to limit the "depth" to which such substitutions are allowed to occur. For example, replacement of (0011) with (110011) creates a new (0011) sequence. Shall it be operated upon again by the rule? If so, recursion will generate an infinite string by repeated substitutions at that site. If not, the depth has been limited. Limiting depth limits the length of the transformed string with respect to the input string.

Recursively enumerable grammars, which can be defined by a finite list of pairs of symbol strings where the partner on the left can be shorter or longer than the partner on the right, are as powerful as universal Turing machines. Tuning the number of pairs of symbol strings, the lengths of those symbol strings, and their symbol sequence complexity tunes the power and character of the grammar. A further "amplification" parameter specifies by
how much and whether always, or on average, substituted symbol sequences are longer than the sequence substituted. Additional grammar rules allow strings to be cleaved or to be ligated. In short, a few simple parameters can be used to specify grammar space. Using them, random grammars within each set of values of the parameters can be chosen and the resulting string dynamics studied.

A simplest approach is this: Use a random set of pairs of strings as the random grammar. Begin with a set of strings, and operate on each of these according to the grammar. Here, however, strings to not act on one another. Stuart Cowan has suggested the same approach.

A more useful approach suggested by Albert Wong, and closely related to Fontana's work, as well as our own origin of life model, is to define grammars of substitutions, gluing and cutting operations, but require that strings contain "enzymatic sites" such that the strings themselves are carriers of the grammatical operators. Thus if the grammar specifies that string "ab" is replaced by string "cddcde", then an "enzyme string" with an "ab" enzymatic site would search target strings for a matching "ab" site, and if found, substitute "cddcde" in the target string at that site. Or the enzymatic string might cut or glue strings as sites. Clearly, this can be implemented in binary strings, with matching as complements or identities.

More complex "machine" and input bundle sets can also be built up by generalizing on the idea of enzymatic sites. Real proteins often cooperate by forming multimeric enzymes carrying out the same or even a succession of biochemical transformations. Here the constituent monomer proteins recognize one another and self assemble within the cell to form the ordered protein aggregate which is the cooperative complex enzymatic machine. Similarly, we might extend the grammar rules to specify how ordered collections of strings self assemble and act as machines or input bundles to yield unique output sets of strings.

The use of grammars is likely to be very important in analyzing the emergence of functional adaptive systems. The Boolean idealization allows the set of all possible next strings to be followed. But it does not readily allow for growth in the numbers of copies of each string, for inhibitory interactions hence competition between strings, and so forth. In contrast, just such features emerge readily in models where strings interact with one another via grammatical rules. I return to this below in considering the implications of these ideas for mutualism, community
structure and economics. In addition, the concept of "sites" on strings carrying out transformations on other strings, the analogue of real enzymatic sites in proteases acting on proteins, will prove useful in thinking about strings encoding multiple transformations. For example, overlapping sites will correspond to strings which carry out many transformations with a minimal content.

A major question is the relation between grammars and Boolean world transition matrices. That is, given the Boolean picture and a mapping of input to output matrices, what equivalent set of grammars does each such deterministic mapping correspond to? Are there some mappings which are not statable as grammars of M strings acting on B strings to yield O strings? A second point is that while all grammars will yield a Boolean mappings of the set of strings into itself it is not yet clear to me that each such possible Boolean mappings is derivable from one or more grammar.

The relation between grammar complexity and the kinds of objects, jets, eggs, fogs, etc. which arise is a central object for analysis. Some points already seem plausible. A simple grammar may be more likely to give rise to finite eggs. A complex grammar may be more likely to give rise only to infinite mushrooms or filligreed fogs. The reason is intuitively clear. The first finite autocatalytic sets found were the hexamer single stranded RNA, and its two trimer substrates, as noted above. The point-point complementarity due to base pairing allows this system to make an exact complement, then itself, in a closed cycle which need not expand out into sequence space. It is possible for this autocatalytic set to remain a 2 cycle, and finite. Once overlapping "sticky" ends and ligation are allowed, this system can give rise at least to infinite filligreed fogs. Now consider a very complex grammar the "punctate" rule. Here, each string has a fixed probability of catalyzing any reaction. The grammar is complex in the sense that, after catalytic interactions are assigned, the set of "sites" in the enzyme which can be taken to act on sites in substrates is indefinitely complex. In due course in supracritical systems under the "punctate" rule, the formation of all strings will be catalyzed, hence this system creates a pea soup. It seems highly likely that the more complex the grammar, the less easy it will be to limit string generation to finite eggs.

The Growth and Asymptotic Form Mutual Information as Strings Act on Strings.
The action of strings on strings to produce strings according to a grammar should build up constraints in symbol sequences in the strings which are produced over time. Such constraints show up in a measure of relations between symbols called mutual information. The mutual information between pairs of symbols $S$ apart is defined as:

$$M(S) = \sum_{a,b} P_{ab}(S) \log_2 \left( \frac{P_{ab}(S)}{P_a P_b} \right)$$

where $P_a$ or $P_b$ is the frequency of value "a", here 1 or 0, in the set of symbol sequences, and $P_{ab}(S)$ is the frequency of symbol value $a$ at position 1 and symbol value $b$ at position 2 at distance $S$ from position 1. $P_{ab}(S)$ is averaged over all pairs of positions $S$ apart in the set of symbol sequences under considerations.

In natural language texts, mutual information, $M(S)$ typically decreases as an inverse power law as $S$ increases, (Li 1990?). Thus, nearby symbols tend to be more strongly correlated than distant symbols.

Consider now a system of 1000 binary strings, each chosen at random among strings length 100. Because the set is chosen at random, the mutual information between sites at any distance, $S$, will be 0. Let the strings act upon one another in a chemostat such that 1000 strings are always maintained in the system. As these mutual interactions occur, the action of strings on one another creates correlations, hence mutual information. Preliminary studies with David Penkower in my laboratory at the University of Pennsylvania indicates that, in fact, in these systems mutual information begins very close to 0 and builds up as interactions take place to an asymptotic form which depends upon the grammar. Typically, simpler grammars appear to lead to higher mutual information. Typically, mutual information builds towards an asymptotic form which is high for small values of $S$ and decrease as $S$ increases.

These preliminary results suggest that the time course in which mutual information is built up to the asymptotic form as a function of numbers of string interactions, and that form itself, give information about the complexity of the grammar itself.

Interestingly, one can envision experiments in which random single stranded RNA molecules perhaps length 100 are allowed to interact with one another. If these mediate ligation, cleavage, and transesterification reactions as do hexamers and ribozymes, then over time the sequences in the system should build up mutual information as a function of $S$. This should be testable by using PCR amplification, cloning, and sequencing of the interacting
RNA sequences over time. In turn, estimates of grammatical complexity are bulk estimates of enzymatic site complexity as RNA sequences act catalytically on one another. Similar efforts may prove useful for mixtures of initially random polypeptides or other potentially catalytic polymers.

Stochastic Generalization.

The model is fully deterministic. Expand it to include random bit mutations in strings to yield stochastic versions of the same basic model. Note that the grammar rules, applied to strings without reference to use of other strings as tools, is the analogue of spontaneous reactions occurring without an enzyme, in the autocatalytic polymer set model. Hence these are the natural form of spontaneous mutations in these systems.

Section IV. Applications to Biological, Neural, and Economic Systems.

Random grammars and the resulting systems of interacting strings shall hopefully become useful models of functionally integrated, and functionally interacting biological, neural, psychological, technological or cultural systems. The central image is that a string represents a polymer, a good or service, or a role or element in a conceptual system. Polymers acting on polymers produce polymers. Goods acting on goods produce goods. Ideas acting on ideas produce ideas. The aim is to develop a new class of models in which the underlying grammar implicitly yields the ways strings act on strings to produce strings, interpret such production as functional couplings, and study the emergent behaviors of string systems in these various contexts. I consider first the implications for biological models.

Part of my own interest in models of autocatalytic polymer systems, beyond the serious hope that they bear on the origin of life on earth and presumably elsewhere in the cosmos, lies in the fact that such systems afford a crystalline founding example of functional wholeness. Given the underlying model of chemical interactions, one an autocatalytic set of catalytic polymers emerges, it is a coherent whole by virtue of achieving catalytic closure. Given the underlying model chemistry and catalytic closure, the functional role played by each polymer or monomer in the continued existence and proliferation of the autocatalytic set is clear. Note that we here feel impelled, almost required to begin to use functional language. This reflects the fact that such a self reproducing system allows a natural definition of the "purpose" of any polymer part subservient to the overarching "purpose", abetted by natural selection,
to persist and prevail. In this nonconscious sense, an autocatalytic set becomes a locus of agency.

Model autocatalytic sets are natural testbeds to study the emergence of collaborative or competitive interactions. We need merely specify how such systems may export or import strings to one another, and we will find in consequence how they come to cope with such exchanges. As remarked above, such interactions are literally what it means for such systems to come to "know" one another. By studying these properties across grammars, it should be possible to understand how grammar structure and well as the structure of interacting autocatalytic sets, governs the coupled coevolutionary structures which emerge. The ways model autocatalytic sets build internal models of one another may well mimic the ways E. coli and IBM know their worlds. In turn, this may well yield insight into the onset of mutualisms, symbiosis, and competition in the biological realm.

Potential Psychological and Neurological Implications.

Artificial Intelligence has harbored a long debate between those who favored models of the mind based on sequential inference as exhibited by sequential computer programs, and parallel processing neural networks. The former are widely used in expert systems, in analyses of linguistic and inferential webs, and so forth. Parallel processing neural networks have remerged more recently as models of content addressable memories. Here a dynamical attractor is thought of as a memory or as the paradigm of a class. All initial states flowing to that attractor achieve the desired memory or class. Hence such systems generalize from attractor to basin. Learning consists in sculpting attractor basins and attractors to store desired patterns of neural activity.

Random grammars and the consequent models of strings acting algorithmically on strings to form jets, eggs, mushrooms, or fogs, seem to be a new and useful marriage of the two classes of AI models. Like sequential rule based models where one action or classification triggers downstream cascades of actions, one string or a set of strings creates downstream cascades of strings. Like parallel processing networks, here many strings can act on one another in parallel to create jets, mushrooms, eggs, or fogs. Unlike the more familiar models, where the couplings among the elementary processes must be defined by external criteria, in grammar string models, the coupling among those processes is defined internally by the grammatical rules which determine how strings generate one another. There is an important sense in which the "meaning" of one elementary process with respect to others is given by lo-
cal production transformations and the global structure, jet, egg, mushroom and its natural dynamics.

Another feature of grammar models is that the set of processes is open and potentially infinite, unlike familiar parallel processing models. Such systems may remain perpetually changing, always out of equilibrium, always adapting, rather than falling to simple dynamical attractors.

It is not entirely implausible that such grammar-string models may prove useful in thinking about the "schemas" by which personality elements are constructed. Consider, for example, the stunning phenomenon of multiple personalities. Typically each "self" has only faint or no awareness of the alternative personalities. The situation is rather like a gestalt shift when regarding a Necker cube. When seen in one way, one literally cannot simultaneously perceive the cube in the second way. The two are mutually exclusive perceptual organizations of the visual world. It seems of interest to consider an "egg" able to interact with an external world as a kind of "self" which knows and organizes its world in some self consistent way. But the same system may harbor more than one "egg", each mutually exclusive of other eggs, each living in its own self consistent world.

Another feature of the image seem useful and may relate "holism" in science to stability of egostructures and "centrality" in the web of string processes. First, consider the Quinian thesis of holism in science. Suppose I hold the earth to be flat, you hold it is round. We perform a critical experiment at the sea shore watching a ship sail out to sea. I predict it will dwindle to a point. You predict the hull will lapse from sight before the superstructure. Your prediction is confirmed. "The world is round, admit it!", you claim in jubilation. "No", I respond, "light rays fall in a gravitational field, so of course the hull disappears first." Quine’s point is that any hypothesis confronts the world intertwined in a whole mesh of other hypotheses, laws, and statements of initial conditions. Given disconfirming evidence, consistency requires that some statement(s) of the premises be abandoned. But we are free to choose which premise we shall abandon and which we shall save, I can "save" my hypothesis that the earth is flat at the price of a very bizarre and convoluted physics. We cannot avoid Quine’s point. Typically we choose to save those hypotheses that are the most central to our conceptual web, and give up peripheral hypotheses or claims about initial conditions. But that very choice renders those central claims very hard to refute, indeed, almost true by definition. Now the interesting point to add is that the hypotheses we choose to save are those which is a graph theoretic sense, are central to the conceptual web. Letting a string process creating an egg, mushroom or other object connected via string ex-
change to an outside world stand as our model of a conceptual framework, that egg entity will have more central and
less central elements. If an "egg" is a "self" knowing its world, preservation of self becomes preservation of the cen-
tral elements in the egg while a peripheral "metabolism" fluctuates into and out of existence. Indeed, one wonders if
the concept of resistance in psychotherapy, a phenomenon familiar in practice if hard to quantitate, can, in part, be
made sense of in terms of preservation of core elements of the egg. One can consistently continue to maintain that
the world is flat despite apparently enormous evidence to the contrary.

Models of Cultural Coherence and Transformation

What did China's leaders know in the summer of 1989? What occurs when an isolated culture come into
contact with a world culture? What constitutes the integrity and coherence of a culture and how do new ideas, myths,
production techniques, transform the culture? Just as it is a vast jump from grammar-string models to models of per-
sonality structure, so too is it hubris to leap to cultural models. Yet the phenomena feel the same. New strings are in-
jected into an egg. It transforms to something different and coherent, even perhaps stable if unperturbed, another egg,
another closed coherent culture. Conversely, modern society is open, explosive, changing, indefinitely expanding in
ideas, goods, services, myths. Have we now become culturally supracritical? Can we construct models in which cul-
tures can be stable Eggs then transform into a different kind of object, a Fog? It seems worth serious consideration.

Application to Models of Technological Evolution of Economic Webs.

Grammar models may prove useful in developing a new class of theories about technological coevolution.
Surprisingly, although technological evolution is thought by many to be a major, perhaps the preeminent factor driv-
ing modern global economic growth, economists lack a coherent theory of the phenomenon. The problem is that the
issue is not merely economic, it is technological. In a sense which requires unpacking, the goods and services in an
economy themselves offer new opportunities to invent yet further goods and services. In turn, new goods and services
drive older goods and services out of the economy. Thus the system transforms. For example, the invention of the au-
tomobile lead to the requirement for a host of other goods and services ranging from paved roads, traffic lights, traffic
police and courts, to oil refineries, gasoline stations, motels, automobile repair facilities, parts manufacturers, emission
control devices. And the advent of the automobile led to elimination of the horse for most transport. With the horse
went stables, public watering troughs, smithies, the pony express, and a host of other goods and services. Thinking of
goods and services as strings in a grammar-string model, injection of a new string engenders many new strings and drives a set of old strings from the structure.

This example states the problem faced by the economist. In order to understand the current "web" structure of the goods and services of an economy, and how that very structure governs its own possibilities of transformation by the invitation to invent new goods which intercalate into the web, transform it, and eliminate other goods, one needs a theory for which goods and services "fit" together technologically.

Economists call such "fitting" complementarity. Thus, the nut and bolt are complements, hammer and nail are complements, and so forth. Complements are sets of goods or services which are used jointly to produce a given other good, service, or consumer product. Substitutes are sets of goods which might substitute for one another in a given production technology or consumption good. Screws can substitute for nails, KCl can substitute for ordinary salt at dinner. The growth of modern economies is dominated by the growth of new technologies. Each new good or service may afford a new niche into which another new good or service can fit. The invention of computers has yielded the proliferation of software engineers, system engineers, and hacks.

It is just such a theory which appears beyond current reach. And it is just such a role that random grammar models may play: If a good or service is modeled as a symbol string, then each grammar and its consequent string transformation rules, amounts to a model of the technological couplings among goods and services. If broad regimes exist in grammar space which yield similar economic consequences, and one regime maps onto the real world, we may attain a model of the technological couplings among actual goods and services and the consequences for economic growth.

A concrete way to build grammar-string models of economic growth is the following: First, specify a grammar by which strings act on one another to produce strings. The set of strings which are a machine, $M$, that jointly act on a string or set of strings to produce an output set, are complements. All parts of $M$ are needed to make the product. Alternative strings, or sets of strings which, as input to $M$, yield the same output set are substitutes. Weaker senses of complements and substitutes arise if output sets which are overlapping but not identical are considered. The machines and their input and output relations are the production technologies of the model economy.
Thus, the transformations specify the numbers of each type of string required as input or machine part to make a specified number of each kind of output string. Thus the grammar implies an input output matrix. To this a formal economic model can add constraints on exogenous inputs to the economy, such as raw material mined from the ground. These might be supplied by a founder set of strings maintained at a constant "concentration". To carry economic analysis further, the utility of each string must be specified. A simple, if arbitrary choice is just the length of each string. Finally, a budget constraint must be specified. This can be the total number of goods and services now in the economy plus the exogenous input from the founder set. Given these constraints, the "equilibrium" for the current economy specified in terms of the linked set of goods and services is that \( \text{ratio} \) of production of all the goods and services which maximizes the total utility of all the goods and services in the economy subject to the budget constraint. That ratio can also be thought of as the price of the goods relative to one another, taking any single good as the unit.

The growth of the economy over time in terms of the introduction of new goods and services can be studied as follows: Start at the current equilibrium, with the current set of goods and services. Use the grammar rules to construct all possible new goods and services derivable by allowing the current goods and services to "act" on one another in all possible ways. This generates all possible new goods which are technologically "next to" those in the current economy. Alternatively, some random or non-random subset of these might be chosen as potential new goods. The "next economy" is constructed containing the potential new goods and services plus all the current goods and services. These new and old goods specify, via the grammar, a new input-output matrix for the economy. The equilibrium of the new economy, as derived from its modified input output matrix, is then assessed. At that equilibrium, some of the new potential goods may "make a profit", hence be produced at a positive rate. Others may make a loss, hence not be produced at a finite rate. Similarly, some old goods will still make a profit, others will now make a loss. The new economy is comprised only of those old and new goods which jointly make a profit. Hence cascades of new goods enter the economy, cascades of old goods are driven from it.

Economic models of this type would seem of interest in a number of regards.

First, they actually model the growth of economies due to the growth in niches afforded by goods to create new goods.
Second, phase transitions occur and may model economic take off. Economies which have too few goods may not be supracritical, hence may never take off. But if the initial economy is more complex, or if several economies come into contact and exchange goods and services, the coupled system may jump from one in which each separate economy makes a small finite jet to a supracritical mushroom which explodes into the space of potential goods and services. Hence, a model for economic take off, even, perhaps, a model for the Industrial Revolution.

Third, the decidability problems in filligreed fogs and other objects imply that it may be logically impossible to deduce that a given good is not ultimately producible from the current technologies. This implies that markets must be incomplete.

Fourth, the same failure of decidability may imply that economic agents must, logically, be boundedly rational. Both these points cut at the core of neoclassical economics, hence may invite its extension.

Tentative Summary

This is a first working draft to state some issues which we must address to build a theory of functional organization and integration. It is my hope that this line, coupled with an investigation of dynamical stability, will yield real insight into the emergence of complex adaptive entities which collaborate in constructing and knowing their worlds. Perhaps E. coli and IBM are governed by similar principles.

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