Building Theories of Economic Process

Martin Shubik
D. Eric Smith

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Martin Shubik
Cowles Foundation, Yale University, 30 Hillhouse Ave., New Haven, CT 06511

Eric Smith
Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501
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Economics concerns phenomena on the interface between materials and processes in the physical world, and strategic actions in cognitive, legal, and cultural realms. A very limited set of questions about the existence of rationally optimal allocations of goods has been asked within the mathematical framework of economic general equilibrium theory. Independently, diverse *ad hoc* models of particular economic systems have been built in operations research, macro-economics, and most recently in the growing field of econophysics. Adequate attention has not been given, up to now, to frameworks for integrative modeling of economic *process*, comparable in generality to the mathematics of equilibrium theory but capable of reproducing the dynamics and contingency structure of physical and strategic transformations, and the institutions that serve them. We sketch a possible form for such a general modeling framework, with particular attention to the control relation of the financial sector to the underlying productive economy. We emphasize the importance of the structure of physical processes to economic phenomena, particularly symmetries associated with scaling and the continuous or discrete passage of time. Finally, as an example of the kind of problem such a framework seems necessary to understand, we discuss the famous Fisher equation for the velocity of money, and its implications for the use of money supply as a control over trade.

I. INTRODUCTION: PHYSICS-LIKE APPROACHES TO ECONOMICS

Economic activity encompasses materials and processes in the physical world, the behavior and choices of individuals, and organization at the social level in such institutions as norms and laws, markets, and government. In principle there is a broad range of observation within which regularities may be found, and attempts to base a quantitative science on those regularities could be conducted in many ways. Historically, starting with the work of Walras [1] and Jevons [2], and culminating in the general equilibrium theory of Arrow [3] and Debreu [4], the mainstream of mathematical economics has sought to define optimal resource allocations and exchange rates (prices), starting from assumed regularities in people's preferences for one consumption profile relative to another.

In the highly abstract existence theorems for optima in this theory, institutions have been regarded as epiphenomena of preferences, whose form is at most a convenient calculational intermediate in deriving prices and allocations from the preferences [5] on which they are assumed solely to depend. The description of processes to either select or reach optimal allocations has been similarly omitted, except in the implicit assumption that, whatever the pre-trade allocations in a society, trade to equilibrium somehow takes place entirely at the final equilibrium prices [4]. Though mathematically sophisticated and elegant, the existence theorems for general equilibria can only be regarded as part of a scientific theory in cases where the mathematical treatment of preferences, and the optimizing function of trade, can be justified at least as approximations. The difficulty of doing so for almost any real market has prompted parallel tracks of research, with different goals and free from dependence on such thorough mathematical representations of preference and the behavior it induces.

One such alternative, the rapidly growing enterprise of *econophysics* [6] (see also http://www.unifr.ch/econophysics) has mostly emphasized statistical regularities in the function of real markets or national economies, and sought to explain these with ideas from statistical mechanics, dynamical systems, and probability theory, in the context of quite explicit process models. Because of their well-defined boundaries, and profusion of (often well-sampled) high-frequency data, financial markets have particularly benefitted from this mode of analysis, as companion essays in this volume demonstrate.

In this article we are concerned with other applications of the methods of physics to economics in general, and the theory of money and financial institutions in particular. Our emphasis is not on the dynamics of particular market models, or on the extraction of behavioral regularities, but rather on the functional roles of institutions that prompt their emergence and may act to stabilize their forms. For these questions the appropriate focus is not on prediction, in either the sense from neoclassical economic theory or that from econophysics process modeling, but rather on the control function of institutions, and the needs for control in the society to enable coordination. Our main premise is that the financial system is the controlling network of the economy. It provides the variations in the constraints that guide the flow of resources and information. Mathematically the influence of many of these constraints is manifested in shadow prices of cash flow limitations. These in turn depend on and clarify the concepts of what constitutes money and credit and how they are issued and destroyed.

A. Prediction versus constraint and control

Many of the limitations of both economic general equilibrium theory and econophysics models can be traced to an emphasis on prediction. The existence theorems of general equilibrium theory were formulated to address the problem of optimizing trade of fungible chattels, so on one hand they assume very little structure on the representation of goods, and on the other hand they assume an extreme specification of people's preferences and a perfect constraint on trade, in order to obtain "the prices of all commodities and the actions of all agents in a private ownership economy" [4]. It has long been recognized that such prices are not generally unique even within the assumptions of the theory, and that even if they are, as real trade is not generally so constrained, the economy need not arrive at any of the so-called "competitive equilibria" [7]. We have argued [8] that this is a real property of the economy, resulting from the irreversible nature of useful trade, and that prediction from preferences is in principle an ill-posed goal, as its mathematical counterpart is for many similarly irreversible processes in physics.

In contrast, the predictions in econophysics, often concerning distributions rather than outcomes, are well-posed goals, and furthermore the emphasis on particular process models makes their validation feasible [9–11]. Such prediction is akin to operations research, which obtains accuracy by restricting its scope and by modeling dynamics thoroughly.

A general framework that separates the modeling of process from considerations of optimality, and thus separates the representation of constraint from the problem of prediction, is the game theory of von Neumann and Morgenstern [12]. Game theory recognizes three levels of representation, which may be independently specified [13]. The first is the definition of *agency* by the introduction of possible *strategic actions* for each of a class of agents. The contingencies of actions, among agents and through the course of play, define the *rules of the game*. The second level is the initial condition for the agents, corresponding to the usual economic notion of *endowments* in a general-equilibrium problem. The third level is the specification of agent preferences and a so-called "solution concept", which tells whether they act independently or have access to means to form coalitions. The rules of the game are logically the most primitive of the three levels, they are the closest to measurement in the real society through the description of its institutions, and they are most naturally cast as the shared constraints on agents, whose endowments or solution concepts may otherwise vary. Emphasizing the rules of the game and the strategies available to its players also provides a natural framework within which to formalize the notion of control.

Without assuming we know either endowments or preferences, we can frame the problem of emergence of economic institutions as one of *enlarging the sets of strategic possibilities available to agents.* Whatever the solution concept in a game, the best solution in a space of possibilities enlarged by the presence of new economic institutions is at least as good as the solution in the subspace of actions possible without the institution. We are, of course, supposing that economic institutions increase possibilities without foreclosing others, at least within the domain of solutions the agents use, but this is a considerably weaker assumption than restricting to particular forms for preferences or a particular solution concept. The problem of enlarging strategic possibility spaces is interesting because it has *structure*; each distinct type of economic institution adds a characteristic collection of options to the society, overcoming an associated limit on agents' abilities to coordinate their actions advantageously. The structure of the institution follows in part from the problem it solves and the type of coordination it enables.

For example, monetary control does not necessarily provide the tools for precise prediction of the economic dynamics of individual behavior, but it can influence profoundly the direction an economy can take. The difference in dynamics, between markets with 10% and with 100% margins required for purchases, illustrates that the behavior causing speculative bubbles can be significantly controlled, whether or not it is understood or amenable to a mathematical representation in terms of preferences.

Agents' strategic options can be controlled precisely because they are limited in the absence of economic institutions, and remain limited when institutions are introduced, though less so. The degree to which a given institution relaxes strategic constraints provides an index of economic control. Thus, for example, the availability of markets and credit makes possible new realms of trade and even of production, but the extent to which it does so depends on the amount and cost of available credit, as well as on the capacity of the markets.

B. General dynamics

More encompassing than the general equilibrium theory, game theory is a highly flexible framework for the modeling of strategic process, and we should specify what are the goals for a general but useful description of economic dynamics. In our opinion usefulness comes from the ability to introduce constraints on economic modeling in a principled way, which interpolates between the almost structureless generality of the general equilibrium representation of goods and trade, and the commitment to specific, and often underdetermined and idealized, models [14, 15]. It is these intermediate levels of specification, often involving abstract concepts such as symmetry, topology, or dimension, which permit the partial specification of physical systems in a way that nonetheless produces falsifiable claims. In this article we consider two aspects of the problem of introducing constraint.

First, economic processes lie at the intersection of physical and strategic transformations. Constraints from the physical world may determine what is possible in an economic strategy space, prior to any further specification by law, custom, or the availability of institutions. The distinction between events and ongoing processes, and decisions involving these, can fundamentally structure strategy spaces, and place constraints on what kinds of preferences it is consistent to assume for agents. Even the simple persistence of a physical object may imply both strategic constraints and opportunities that continuously unfold, imprinting the properties of the object on the rules of all possible games involving it. We therefore seek to formalize the ways properties of objects, and in particular topological considerations like the possibility to take actions only at discrete intervals or continuously through time, can be consistently enforced in economic modeling.

Two examples of other areas where the needs for consistency and hierarchical specification must be met in an open-ended (or "generative") system are ontology construction [16] and the familiar domain of formal language specification [17] (as in the definition of computer programming languages). Common features of both are systems of *types*, which restrict rules for transformations and value assignments, and hierarchies of *classes*, which allow partial specification of abstract collections to be systematically refined to particular instances. One goal of a general theory of economic dynamics could be seen as a formal language for economic modeling, in which types reflect the physical and legal constraints on objects and strategic actions, and classes reflect categories of function and relations among objects of different types. Even a cursory specification of such a formal system is outside the scope of this article, and will be presented elsewhere [18] (examples of a few type/transformation pairs are given in Ref. [19]).

A general dynamic system for economics does not serve the same function as either an object-oriented programming language, or an ontology, and will be different in key respects. One on which we will focus here is the need for economic equations to satisfy requirements of *dimensional homogeneity* and to reflect *scale invariances* when appropriate. In physics these two requirements are the basis of methods of *dimensional analysis* and the foundation of sensible quantitative analysis. Symmetries under scale transformations also provide a way to distinguish continuous from discrete topologies, and thus events from processes. While the importance of dimensional analysis to economics has been advocated in general [20], and demonstrated to predict interesting scaling relations in special cases [11], a system for operationalizing the concept of dimension for general economic problems does not yet exist.

We will apply these ideas, in restricted form, to familiar and fundamental problems of economic organization. In particular, we relate the economic concept of capital to the dynamical concept of catalyst, and address how capital structures economic possibilities. We also ask what about certain goods or legal papers causes them to take on the role of monies, and to what extent "money-ness" reflects the existence of a dimension. Finally, we address two closely-related problems of long-standing interest and debate in the representation of preferences: when do people's trading behaviors assign an effective money value to their preferences, and when can this money value be used to represent an aggregate economy as having preferences equivalent to those of an agent?

II. DETERMINANTS OF THE STRUCTURE OF AN ECONOMY

Any economy is structured by the substrate of its physical products and processes, and by legal, normative, and informational constraints on the actions of its agents. Little production in human society since the advent of agriculture has resulted from the sole input of labor and natural resources; almost all productive processes use intermediate goods whose function is to convert labor and resources into other types of output. These goods are called *capital goods*. They require resources to produce and often to maintain or dispose of, but while they exist they convert other resources without themselves being consumed. This qualitative distinction between capital goods and primary resources causes capital to have an amplification effect on production that primary resources do not have. Capital goods are the catalysts of physical transformation underlying an economy, and much as the structure of chemical and biological systems is largely determined by their catalysts [21–24], so the structure of production is largely determined by the kinds and amounts of capital goods available to an economy.[1]

Financial institutions serve a purpose similar to capital goods, with respect to the coordination of agents' strategic actions. Markets aggregate and disaggregate the joint actions necessary for trade (and often the process of negotiation) into collections of individual strategic actions that can be taken independently and still rationalized. Monies overcome problems of search for compatible trading partners, and provide rational substitutes for trust in

[1] The concept of "catalytic" function is of course an idealization, perhaps better and certainly more intuitively approximated in chemistry than in much of economics. Chemical catalysts may be completely unchanged in reactions, or may be decomposed by random fluctuations, so they "depreciate" only on the average. Economic capital stock, in contrast, can depreciate systematically and deterministically. If depreciation is considerable the distinction between catalyst and material input to the reaction may cease to be a useful one.

trade [25, 26]. Bonds, stocks, and derivatives overcome problems of aggregation of ownership claims, and segregate different aspects of risk to be purchased separately. Both trade and resource allocation in an economy are determined largely by which institutions exist and how they function. Like capital goods, the institutions themselves are not consumed in providing such services, though they require resources to maintain, and to convert into their respective outputs. For each such class of instruments, we wish to formalize the following questions in terms of enlargements of strategy spaces: Why do they exist and what purposes do they serve? Is there a basic number of these instruments that will support any modern economy? Are there any natural bounds to erecting structures of paper on paper as population and the complexity of the society grows?

We illustrate in the following subsections how these questions can be organized in terms of a system of types and transformations, capturing in the process some interesting consequences of dimensionality.

A. The existence of dimensions

The concept of *dimension*, in the sense of a type of measure or the units for measurement, is one of the most useful in physics. Dimensional analysis works because the number of physical dimensions is limited, yet which limit is relevant depends on the problem under consideration. There are only three "fundamental" physical dimensions, which may be chosen to be length, mass, and time, and many other "derived" dimensions that may be written as products of powers of the former, such as velocity, momentum, energy, viscosity, action, and so on. In many problems more abstract quantities like temperature, entropy, or chemical concentrations can be regarded as defining dimensions. It clearly misses the point of the usefulness of dimensionality to try to derive economic dimensions from physical ones, or to limit ourselves to the dimensions that might be so defined. Yet in economics as in physics, not every quantity representable by a number has dimension. Therefore it is essential to set forth criteria by which something qualifies as a dimension, and for the independence of dimensions from each other.

For our purposes, any dimension corresponds to a kind of *substitution relation*, defined in terms of a scale-independent process, and admitting sub-division. We will use examples to make this intuitive, rather than try to formalize it further. Thus, we could define 'length' from the process of separating two objects in space, by placing something between them. The operation makes sense for objects separated by arbitrary amounts (perhaps subject to some upper and lower limits), and length is sub-divisible because 100 cm-sticks stacked end to end are in all respects substitutable for one meter-stick for the purpose of separating two objects. Similarly weight is operationalized by balancing objects on a scale, and weight is substitutable because sixteen one-ounce weights can be substituted for one pound-weight.[2] Weight and length are independent, because there is no known physical process by which the balancing of a scale can in all cases be converted to the separation of objects via the mediation of that process.

Time will be an obvious dimension in both physical and strategic realms, operationalized by using the ticks of a clock to fill delays between events, and subdivisible because fast clocks ticking in sequence can be substituted for slow clocks. Time is especially important because it can convert other quantities with dimensions into their corresponding *rates*. We will argue that it is natural for money to be treated as a dimension, and from the way *utilities* [4, 27] are used to represent preferences in intertemporal decision problems, they are often implicitly (and perhaps less naturally) assumed to have dimensionality. Interest rates, which have units of $(\text{time})^{-1}$, convert monies into rates of debt service, a fact that will be important to us when we consider the problem of aggregation of agents in an economy. Similarly, if utility is regarded as a dimension (at least within the decision space of a single agent), integrals over time will be needed to relate *utility rates*, which are natural for the evaluation of processes, to utility changes at events. In considering financial market function, we have used densities of order arrival in both time and price as dimensions [11], to predict nontrivial relations among these and measures of liquidity and volatility.

Our definition of dimensionality in terms of scale-invariant substitution relations emphasizes that dimensional homogeneity of equations is a property related to the simplification of systems that have scale-invariant dynamics, and it is no coincidence that the same dimensional units should appear in both relations. Homogeneity is the requirement that quantities expressed on either side of an equality must be measured by the same kind of substitution relation, and that they are expressed numerically in relation to the same units. Many economic quantities, such as strategic actions, and especially utilities of different agents (if such can be said to exist), are not generally substitutable or subdivisible in any regular way, and thus the representation of aggregations of agents by the same kinds of preferences or strategies as are used for individuals is not generally possible. We will adopt the convention that a numerical measure cannot be used unless it is either a pure number (such as a counting index) or is assigned a dimensional unit of measurement.

[2] An equivalent operation utilizing momentum rather than gravity operationalizes the dimension of 'mass', for which we could have used a measure of mass that is familiar, the gram.

B. Events and processes, stocks and flows

The most basic qualitative distinction among physical (or strategic) transformations is between those that naturally occur as *events*, and those that occur as ongoing *processes*. Buying an automobile or eating a pint of ice cream are for many practical purposes events, while the provision of electric power or the the payment of interest on an outstanding loan are processes. Perhaps even more interesting, if the loan is of unspecified duration, the decision at each moment to repay the principle or to go on paying interest is itself a process.

The basic distinction between events and processes is immediately obscured by the common practice of modeling intertemporal economic phenomena in terms of "periods", whose duration is unspecified or otherwise unconstrained. Events are naturally framed within periods, but processes require something more. Processes can be represented in discrete models, but they require that only an *equivalence class* of discrete models refer to reality: the periods in each model must be given a size, and a scaling relation among period size and other economic quantities must be specified to relate models with different periodicities. A continuous economic process is one in which nothing about the period size appears as a characteristic scale factor in economic observables.

A similar distinction is recognized in economic theory, between *stocks* and *flows*. If stocks represent persistent goods, flows may represent their rates of change. Alternatively, flows may represent the input of labor or the rendering of services (by labor or machinery). As the delivery of flows is the same kind of continuous process as the passage of time, in consistent economic modeling there cannot be dimensional homogeneity between stock and flow variables without the specification of characteristic times. The strategic options associated with flows are also different from those associated with stocks, and if preference relations are to be defined, preferences for flows and stocks must respect the differences in the strategy sets on which each is defined. Thus the distinction between events and processes, or between stocks and flows, introduces the most important and universal dimension into economics: that of *time*.

We might thus begin to build a formal system for economic modeling by introducing variable types called **Labor** and **Service**, as one defines a subset of properties of a programming language according to its distinction of integer from floating-point numbers. **Labor** and **Service** correspond to variable types at the abstract level of *classes*, in that there are some properties common to all **Labor**s and **Service**s, but others that must be specified by subclasses and even particular instances. We will use boldface to refer to classes, whether of variable types or other objects.

Physical services are easy examples to understand. Delivery of electric power of a certain voltage and frequency (which may also be continuously indexed) is an instance of a **Service**. It happens to have the physical dimensions of power, and can be accumulated over time to another dimensional quantity (the energy delivered), whose dimension is physical energy. **Service**-type variables may be inputs to production, or they may directly open new strategic choices to agents, which have utilitarian value. We distinguish these from **Labor**-type variables only so that the latter imply restrictions on agents strategic options, as the promise of an office worker to be at work between 8:00am and 5:00pm denies him the option of being elsewhere through that time.

The continuity of time for both labors and services is reflected in the fact that they confer, independently at each moment, either strategic options (to consume or to dissipate) or constraints (requirement to stay somewhere or do something rather than to leave or fail to do). If services are accumulated, the amount of the accumulation over a time interval Δt is proportional to Δt in the limit $\Delta t \rightarrow 0$. The use of what we might call "virtual accumulation" of services induces a scaling relation on discrete-period models, and leads to a set of transformations that give meaning to capital goods. Thus if a **Labor**-type input yields a **Service**-type output (laundering clothes), the amount of labor consumed and service produced in each period is proportional to Δt. Note that many markets, such as continuously-clearing double auctions, yield clearing as a **Service**-type variable, converting agent bids and offers that may be continuously placed into clearing contracts (or priorities on a waiting list) with pre-specified properties, as long as the capacity of the clearing system is not exceeded.

C. The catalytic function of capital

Because economics is an activity of living agents, even so simple a transformation as the persistence of an economic good through time may be an active process. Therefore in a system for modeling economic dynamics, it may be appropriate to require the modeling environment to continuously enforce contingencies, such as the persistence of a stock on the continuous delivery of flows. A simple physical example is the requirement of ice cream for refrigeration if it is to persist as a good delivering a certain kind of consumption. Nowadays the refrigeration itself is a service delivered by a machine, which in its turn requires the input of electric power, as well as a protected space and occasional maintenance, and ultimately disposal.

A rich type space is possible for goods that require or deliver services, and in turn may be consumed or may become factors in production of other goods. Here we will consider only one such class, which we denote **Capital Goods**. We might identify interesting subclasses of **Capital Goods**, like refrigerators, by a service they deliver (cold preservation) and another that they require (electric power), either to provide their service outputs or even to persist. Suppose that X is the name of an instance of an input service (a type and amount of power) and Y the name of an output service (a temperature and volume of cold preservation), and that we denote the subclass of **Capital Goods** for which these are input and output by $\chi(X, Y, \tau)$. We have also characterized the class by a remaining "service lifetime" τ, measured from the present, after which it will be transformed into a different good (for instance, a dead refrigerator requiring disposal).

It is conventional in general equilibrium economic modeling to represent any collection of goods (for instance, the collection owned by some agent) as a vector, whose basis elements are measures of each type of good, and whose scalar coefficients are real numbers multiplying the basis elements. Just as our types can have dimension, we can regard them as basis elements in a vector space describing the physical state of the world, only now there are distinct kinds of basis elements for stocks and flows. We may represent all states in terms of stocks in a discrete-time model, as long as the scaling of stocks representing accumulations of flows respect time continuity.

We may then capture the interesting transformation properties of capital goods by representing their action on the class $\chi(X, Y, \tau)$, hence on all of its subclasses and instances, in conjunction with the consumption of service X and the production of service Y. The transformation on the basis elements, represented as a map of vectors, induces the transformation on a general goods space by multiplication with appropriate real coefficients.

Over any short period Δt, the modeling environment is required to perform the replacement

$$(X\Delta t) + \chi(X, Y, \tau) \rightarrow (Y\Delta t) + \chi(X, Y, \tau - \Delta t), \quad (1)$$

(and if it cannot do so, due to the unavailability of X, to transform the capital good or its service in some specified way). Eq. (1) says that an amount of accumulated power $X\Delta t$ (dimensionally, an energy), and a refrigerator of service lifetime τ, is transformed into an amount of cold preservation (volume and time) $Y\Delta t$ and an older refrigerator of service lifetime $\tau - \Delta t$, and that this transformation is the only admissible representation of the passage of time for this class of goods. If $\chi(X, Y, \tau - \Delta t)$ differs significantly from $\chi(X, Y, \tau)$, the good may be said to depreciate. If we consider a multiple s of the refrigerators represented by $\chi(X, Y, \tau)$, it similarly transforms $sX\Delta t$ into $sY\Delta t$ per interval Δt, so both sides of Eq. (1) are multiplied by s.

If, relative to the timescales of interest, $\tau \rightarrow \infty$, Eq. (1) describes the continuous conversion of services $X \rightarrow Y$, contingent on the presence of a good in the class $\chi(X, Y, \tau)$ capable of that conversion. The stipulation that this transformation apply for *any* sufficiently short Δt identifies X and Y as services, and aging as a process consistent with a continuous topology for time. Such a continuous but contingent conversion of services is the defining characteristic of *catalysis*. A model that specifies the possible classes $\{\chi(X, Y, \tau)\}$ of capital goods, and procedures and costs for the creation of each, defines the physical constraints on transformation of services in an economy. As services can equal the time rate of change of inventories of stocks, this conversion amounts to a process model for production as well.

In Ref. [19] we provide related discussion for consumable goods, including the notions of durability and perishability.

D. Markets and monies

The cultural and legal concept that creates an economy from people and physical goods and processes is a system of *ownership rights*. Private ownership is the simplest system to formalize, because it associates with every good or process an agent with rights to dispose of it. However, the nominal simplicity of private ownership rights displaces the complexity of social coordination onto the system of *contracts* that enables agents to transfer ownership. The two most severely violated assumptions of general equilibrium economics are of the existence of *complete* and *costlessly formable and enforceable* contracts. The failure of complete contracts leads to the economic notion of "externalities", consequences of economic activity that are not part of the bargaining process during contract formation. The task of providing and enforcing contracts at acceptable cost is served by economic and legal institutions. We will consider only two here: markets and monies.

Trade is intrinsically a joint strategic act by two or more agents. If the agents are asymmetric, like a government in relation to one of its citizens, the joint act of trading may be consistent with the notion of agency, because the government may act unilaterally, while the citizen's act is contingent. Voluntary trade among equals involves a breach of the notion of individual agency, because each agent's action is contingent on the other's. Markets work around this breach by disaggregating the joint action into independently performed individual actions, such as bidding and offering. Agents unilaterally relinquish either ownership or control to markets, and in exchange markets implement pre-specified algorithms for converting bids and offers into *clearing contracts* among two or more agents, to which the agents are then bound by law or custom. The unique restriction on markets is that they fulfill the pre-specified algorithm for any possible instance of agent bids and offers, making possible the rational evaluation of unilateral action by agents. We return in Sec. III to the formalization of these strategic disaggregation mechanisms.

The one element common to all market functions is the submission of *bids* [28], and with these the notion of "money-ness" as a new, quintessentially economic, dimension arises. Markets per se only overcome the problem of creating joint strategies; they potentially leave unaddressed the problems of search for suitable trading partners, overlap of the offered and desired goods of buyers and sellers (known as the "double-coincidence of wants" [2]), and exchange ratios defining acceptable prices to both parties.[3] The goods that historically have become monies overcome these problems.

As a type (not yet implying dimensionality), a *money* is accepted as a bid in most or all markets in an economy, giving a star-like shape to a graph of goods in trade [14]. Near-monies, such as bank credit, may be acceptable in a large subset of markets. Universal acceptance simplifies problems of search, and overcomes the failure of the double-coincidence of wants. A formal type specification of a money or near-money might include as arguments the set of markets in which it is accepted for bids.

The other general features of monies are *divisibility* and *interconvertibility*. Salt, tea, gold, and government-issued paper monies are all by their nature arbitrarily divisible, and all have served as monies at various times. Those that qualify as monies in an economy are also substitutable in some ratio as bids, in all markets that accept them. The operation of bidding at market, together with divisibility, operationalizes "money" as a dimension, while the conversion rate permits the specification of each type as a particular *unit of money* (e.g. **ounces of gold** or **U. S. dollars** as types). As the regulatory problems of bimetallism illustrate, exchange rates between units of money must generally be determined by the dynamics of an economy, so in a formal system the only requirement is that *the operations of subdivision and unit conversion commute at any instant of time*. Thus, the price (conversion rate) for ounces of gold to dollars must be independent of the amount converted if both are to qualify as units of money.

The criteria of money-ness are often only approximately fulfilled, as are the recognized functions of money as medium of exchange, store of value, or numéraire in exchange. Thus only one good may qualify as an ideal money in a model economy, with other goods inheriting approximate function as monies through divisibility and market-determined conversion rates (as in the "money-markets" for short-term debt obligations).

E. Strategy sets and utilitarian valuation

Up to now we have referred to the mathematical representation of preferences, but not described how it is conventionally performed in either general equilibrium theory or game theory, what are its pitfalls, or how it might be constrained in useful ways. In general equilibrium theory and in non-cooperative game theory, preferences are represented by *utility functions* [4, 27, 29]. In the most general case these are ordinal relations, giving for any two possible consumption profiles, a *preference relation*: either that one is preferred to the other, or that the two are equivalent. In conventional models, where consumption at different times or under probabilistic states of nature are compared, ordinal relations are usually proposed at each time or under each state, in which case to be compared across times or across states, their *cardinal* relations must also be specified. Utility in economics has been formulated as a *potential function*, meaning that the equivalence classes of the ordinal relation provide a so-called *foliation* of the space of consumption profiles (a set of non-crossing "leaves" that fill the consumption space), or more stringently, that the difference in the cardinal utility between any two profiles can be computed as the sum of utility differences along any path in the consumption space connecting the two profiles.

The existence of a utility representation of preferences is a notorious problem in economics [30, 31], and in most cases where utilities can be said to exist, it is probably best to regard them as a parametric description of habitual behavior over short times, conditioned by past experience in a technological and cultural setting.[4]

It is conventional in general equilibrium modeling to write goods or services directly as arguments to utility function, as if goods "have" intrinsic utility to a given agent. In a general dynamic model, we find it logically clearer to let the physical characteristics of goods, and legal conventions such as property rights, create *strategy sets* for agents, and then to let a subset of strategic acts be arguments to utilities. This somewhat complicates the representation of valuation of goods (rather than ice cream's "having" utility, ownership of ice cream creates the need to preserve it but offers the option to consume it, and the act of consumption may have utilitarian value), but it seamlessly allows treatment of non-utilization of goods or services (the ice cream may be allowed to spoil, or a refrigerator may go unused), as well as placing trade and consumption on an equal footing as strategic options (where only the latter may have direct utilitarian value). Taking strategic acts as the only primitive arguments of utilities emphasizes that the "utility of goods" in general equilibrium modeling is a derived concept drawing from restrictions on the strategy space, such as total utilization.

Nonetheless, important questions arise about the properties of these general equilibrium utilities when they exist. In particular, supposing that agents fully utilize goods and consume services as part of their solution concepts, and so effectively "have" utilities of those goods and services, when do their behaviors in a market assign a natural money valuation to the goods (a so-called "money-metric" cardinalization of the utility function [27])? Moreover, when does this money-valuation correspond to the price of goods or services in a recognized money, so that the money-wealth agents would

[3] Indivisible cows, cars, or factories are simply not usable as either bid or offer in a variety of small-scale trades.

[4] People do not have inborn utilities for shoes, bread, and life insurance, though in a particular setting where these are conventionally available, they may acquire and behave according to utility descriptions.

exchange for the ability to trade to an optimal allocation is a measure of the welfare gain to the society from trade [8, 25, 26]? When the institutions that support trade (markets, courts, police) are realistically modeled, with costs of labor and resources to create, operate, and maintain, the money-value of trade must exceed their cost of operation if they are to persist in a rational allocation of resources by the society. Finally, when does the ability to exchange money cause the whole collection of utilitarian agents in a society to behave, in the aggregate, as a single agent also having a well-defined utility?

The last question is known in economics as the problem of *aggregatability*, and it has been shown [32] that the most general economies that are aggregatable have all optimal allocations differing only by exchange of a fixed kind of consumption profile among the agents [8], which serves as a wealth measure. However, this wealth measure is only a standard "money", defined independently of the particular endowments of the economy in other goods, if all the agents have utility functions that can be put in the cardinal form $U_i = x_i^0 + u_i\left(x_i^1, \ldots, x_i^n\right)$, where i is a subscript naming the agent, and x_i^j is that agent's consumption profile of good j, in some vector representation for the space of goods. Utilities of this form are called *linear-separable* or *quasi-linear* in the good x_i^0, which is the natural money in the economy because it measures of value of changes in the utility function u_i under the least favorable trades the agent will voluntarily accept.

Linear-separable utilities have been recognized as making possible these strong forms of aggregate representation, but have been regarded as too restrictive [27, 29] to be assumed for general economies. We therefore ask whether general features of the space of goods or decisions may require linear-separable utilities. The most natural such feature concerns inter-temporal lending in continuously available markets.

F. Intertemporal utilities

In an ordinary intertemporal consumption model, time would be divided into periods of length Δt, and receipt of a service X in an amount s_t at the interval with index $t = k\Delta t$ (k and integer) would be assumed to imply consumption of the service. Thus a "virtual accumulation" of the service $s_t X \Delta t$ would be consumed in the interval indexed t. To compare consumption trajectories s_t a utility would be taken of the form

$$U[s] \equiv \sum_{k=0}^{\infty} u_{k\Delta t}(s_{k\Delta t} X \Delta t) \, . \tag{2}$$

We use square brackets to denote dependence on a function s, which represents a whole history of consumption. The functions $u_{k\Delta t}$ are cardinal utilities of consumption acts at each time, and contain whatever discount structure is assumed in the problem. (The conventional economic assumption in building such models is that $u_{k\Delta t} \propto e^{-\beta k}$, where β is a uniform discount per period, and $u_{k\Delta t}$ is taken to be a utility of all forms of "consumption", with no distinction of goods from services.)

We now ask what class of utilities can reflect a decision structure with the same continuous topology as the passage of time. In other words, while the agent may consume the service over extended periods of time, he has the option to cease doing so at any instant, and must thus be able to evaluate trajectories s_t that change abruptly. Of particular interest will be the trade of debt in continuously available markets. If the market defines an interest rate ρ which is a function of time, then over each short interval the debt service on some borrowed principle P is $\rho P \Delta t$, where ρP is an instance of a service, because ρ has dimensions of a pure rate. We may take ρP to be the service $s_t X$, with X a rate of payment with some fixed normalization, and s_t absorbing the time dependence of the agent's choice of P and the market's determination of ρ.

The question is what dependence $u_{k\Delta t}$ can have on its argument. In any expansion in powers of $s_{k\Delta t} X \Delta t$, exponents less than unity cause a divergence of the utility over a finite interval of real time if $\Delta t \to 0$, while exponents greater than unity cause the utility to vanish if $\Delta t \to 0$. Therefore the only scaling with a regular limit is $u_{k\Delta t}\left(s_{k\Delta t} X \Delta t\right) \propto s_{k\Delta t} X \Delta t$.

If we suppose that a collection of services $\{X_i\}$ (of which payment or receipt of debt service is only one) is consumed with histories $\{s_{i,t}\}$, and that additionally there is a collection of goods $\{G_{j,t}\}$ held through the interval, which effectively confer utility within the solution used, the most general utility with a continuum limit becomes

$$\begin{aligned} U[\{s_i\}, \{G_{j,t}\}] &\equiv \sum_{k=0}^{\infty} \Delta t \sum_i s_{i,k\Delta t} X_i u_{i,k\Delta t}(\{G_{j,t}\}) \\ &\to \int_{t=0}^{\infty} dt \sum_i X_i s_{i,t} \, u_{i,t}(\{G_{j,t}\}) \\ &\equiv \int_{t=0}^{\infty} dt \, u_t(\{s_{i,t}\}, \{G_{j,t}\}) \, , \end{aligned} \tag{3}$$

where we refer to u_t as a *utility rate*. Each basis vector X_i for a service appears linearly in u_t, along with its consumption rate s_i, and an arbitrary function $u_{i,t}$ of the stocks held.

If there is a consumption event ΔG_I that can occur at any particular time t_I chosen within a continuous interval, we may incorporate that utility of consumption into the density u_t by taking

$$u_t \to u_t + \delta(t - t_I) \, \Delta G_I u'_{G_I}(\{G_{j,t_I}\}) \, , \tag{4}$$

where $\delta(t - t_I)$ is a Dirac delta function, and $u'_{G_I}(\{G_{j,t_I}\})$ is the marginal utility of consumption by ΔG_I against the background $\{G_{j,t_I}\}$. Integration over the correction in Eq. (4) adds a finite amount to U, in keeping with the event character of consumption of goods. Thus we obtain as a general modeling constraint that, even within the general-equilibrium-style valuation of goods and services, service streams can only be valued by utility rates, and only the integrals of utility rates can be compared to utilities.

Now explicitly represent a history of debt service payments $\rho_t P_t$ as $s_{i,t} X_i$ for the index $i = 0$, and consider problems in which agents must compare the value of payment or receipt of debt service, continuously in time, against the value of other service streams $s_{i,t} X_i$ for $i > 0$. Then the intertemporal utility takes the form

$$U[\{s_i\}, \{G_{j,t}\}] \; \rightarrow \; \int_{t=0}^{\infty} dt \left\{ \rho_t P_t \, u_{0,t}(\{G_{j,t}\}) \right. \\ \left. + \sum_{i>0} X_i s_{i,t} \, u_{i,t}(\{G_{j,t}\}) \right\} . \quad (5)$$

$u_{0,t}(\{G_{j,t}\})$ is the marginal utility of debt service at each instant of time, in the context of the goods held by the agent at that time (generally including P). Except for the presence of this marginal utility, Eq. (5) has linear-separable form in coefficient $s_{0,t}$ of service X_0, purely as a reflection of the continuum limit. If $\{G_{j,0}\}$ is regarded as an initial condition[5], so that only subsequent $\{G_{j,t}\}$ are consequences of strategic choices, the normalization $u_{0,0}(\{G_{j,0}\})$ may be set to unity, in which case the utility is linear-separable in the debt service at $t = 0$ (in a variational sense; in a period model the debt service over a discrete interval Δt is the linear-separable argument).

Linear-separable utility in the initial value of a debt service is a natural consequence of continuous-time decision making, but since the debt service is proportional to $P_{t=0}$ in a market setting, the linear-separable quantity is naturally also a component of wealth. Thus we are not proposing that people naturally have a linear utility of wealth, but rather that the conventional notion of rational inter-temporal decision-making in the context of continuous-time markets for debt makes this the only consistent representation of preferences.

Not all wealth-changes respect the topology of time in this manner. The most familiar cases that do not are stochastic returns uncorrelated over short time intervals, the conventional model underlying investment return in efficient financial markets. Such returns, under scaling of the correlation period $\Delta t \rightarrow 0$, converge to a Wiener process, with everywhere infinite derivative of the return with respect to time. Thus it is explicitly the presence of markets for non-stochastic payments of debt service that motivate quasi-linearity in the initial rate of payment.

[5] The choice to treat the endowment of goods at time $t = 0$ as a boundary condition not subject to strategic action depends mathematically on how the continuum limit is constructed. The treatment we have adopted is economically reasonable, as markets for material goods are often less liquid than financial markets.

G. Bonds and stocks

Intertemporal loans may be secured with collateral or debt obligations, a purely legal form of contingent constraint on the future strategy set of the borrower. Such loans make possible a limited transfer of wealth, whether because default penalties have limited severity, or because the transfer of ownership of complete goods between agents limits liquidity. They are thus often suited to small-scale borrowing by individuals, but not to the large-scale aggregations of wealth needed to create firms. Bonds and stocks are introduced to overcome these limitations, as well as to separate certain divisible rights of ownership from the decision-making that controls the use of capital, and to separate components of risk in different kinds of contingent claims.

General dynamical modeling of bonds and stocks requires the introduction of the **corporation** as a type of agent, different from the utilitarian individual. Corporations have agency in order to own and dispose of property, but they differ from *natural persons* in that corporations are *owned* by other agents. Rather than having open-ended strategies only circumscribed by law (the most that is realistic in the modeling of natural persons), or utility functions, corporations must be supplied with explicitly defined strategic options, and a process that converts strategic acts of the owners into strategic acts of the corporation (similar to "methods" in object-oriented programming). It is natural to let the strategy-generating process be a game played by the owner-agents, but we do not pursue further formalization here.

Stocks are a type of contract that assigns ownership rights of a corporation to a collection of agents. The sale of stocks bestowes on the buyers new strategic choices created by the game that controls the corporation, among which may be the liquidation of the corporation and sale of its assets in some pre-defined market. Stocks generally also assign a *contingent claim* to proceeds in the event of liquidation, though these may be contingent not only on the context of the liquidation, but also on the payment of debts.

Bonds are a different type of contract for a contingent claim, without ownership rights. They may also promise interest as a service, and repayment of a principle or "face value", and introduce default on either obligation as a condition requiring liquidation. Because bond obligations are limited, while the liquidation value of stock is not, bonds obtain their value by assigning first priority to repayment of their obligations in the event of liquidation.

Bonds thus minimize down-side risk while limiting up-side reward, while stocks do the complement. Both of these instruments are readily formally specified as types, though the specification is more complex [18] than those we present here. An important property of bonds is that the principle and interest obligations of a bond issue, which are generally denominated in money, are essentially arbitrarily divisible.[6] Short-term bonds, whose face value is a stock variable like money itself, readily take on properties of monies, and are key components in money markets. Longer-term bonds, whose interest streams (flows) must be valued against their purchase prices (stocks), may qualify as near-monies in non-stochastic environments, but progressively lose this association as interest rate fluctuations cause the value of a determined interest stream to vary.

We have omitted the treatment of risk in this minimal survey of types, which is a subject in itself, both for physical quantities and for utilities. We note, however, that the legal definition of economic rights is based on realized outcomes, and therefore contingent though not generally predictive.

III. STRATEGIC PROCESS ANALYSIS

The general equilibrium model of a production and exchange economy was studied abstracting out both financial instruments and commodity and service flows. The mathematization of Arrow [3], Debreu [4] and McKenzie concentrated on prices and quantities of goods and services in the physical economy, supposing that the physical transformation properties we have described above could transparently be both modeled and evaluated into an indefinite future from the time of contracting. The legal constraints on money and financial instruments were irrelevant for those purposes.

In a world with complete contracts and without transactions costs the financial structure would not matter to the equilibrium analysis. There could be vast domains of indeterminacy involving trade in the financial assets. However once the micro-economic details of process have been spelled out and it is realized that every process consumes resources it is natural to assign resource consumption to the construction and maintenance of markets, monies, and laws [25, 26]; thus the need to specify transaction costs is not merely an institutional detail but a logical necessity of process description. Furthermore if the resource consumption is relatively small in comparison with production processes his implies that slight changes in relative costs could cause large changes in utilization.

The counterpart of the physical process language we have suggested above, for strategic action, is provided by game theory. Catalytic functions in the strategic sector arise from the creation of joint strategies, as by markets, and from evaluation and control often associated with financial instruments, which disaggregate different elements of risk. Games may be specified in so-called *extensive, strategic, and coalitional forms* [13]. The first gives an explicit description of the sequence of possible events and all strategic contingencies, and is readily integrated with physical process models. The second provides a compressed (hence non-unique) description of the structure of independent strategic actions and the payoffs that accrue to them, and may be generated by post-processing of a combination of extensive game and physical process specification. The third assumes negotiations, side payments, enforcement, or other mechanisms for forming coalition strategies from individual strategies, and is appropriate to a meta-analysis of a complete process specification, or for the formation of upper and lower bounds on the requirements of money and financial services in cases where the external influences on coalition formation cannot be well estimated.

A. Levels of game-theoretic analysis

As the real economy is in a sense a playable game and not an abstract problem of equilibration, it is natural to require the same in models. A specification of trade mechanisms at the strategic level results in the *strategic market games* [28]. They are more fully defined than the general equilibrium models, providing a full specification of information conditions and payoffs in all states of the game. They are more institutional in the sense that items such as price formation must be specified. These games can be simulated or utilized in gaming experiments as well as being analyzed for their non-cooperative equilibria.

A simple consideration of the price formation mechanisms shows that for procedures as simple as department store pricing or English auctions or the pricing of speciality steel, it is possible to specify millions of different models and it is scientifically difficult to select the appropriate sensitivity analysis. The local laws of motion require institutional detail. There is, however, a next step to escape from non-process statics, which is to utilize the strategic form of the game in the formal mathematical specification of what we may call a *minimal institution*, a model that is sufficiently rich to display the basic properties of the instruments and institutions involved, yet which cannot be further simplified without losing the properties being considered. It has been argued elsewhere [28] that there are only three minimal price formation mechanisms. These have the technical property that the strategic and extensive forms of the game coincide. These minimal strategic representations can be reasonably well analyzed mathematically without having to become enmeshed in institutional detail.[7]

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[6] Sometimes mutual funds that invest in bonds are required for very small-scale aggregation and disaggregation.

[7] In contrast with *Strategic Market Games*, we may use an approach based on *Market games* and cooperative or coalitional

Some structural analysis of trading models is presented elsewhere [28](vol. I), [14, 15, 25, 26], emphasizing the effect of differences in market structure and money type on the costs and outcomes of non-cooperative strategic equilibria. Here, we attempt to capture some aspects of the flexible and sensitive evaluation and control structure of the web or skein of financial instruments, in a partly aggregate representation. Attempts to formalize these issues without a structured process description have led to conventional descriptions whose meaning and usefulness is unclear.

B. Questions about economic control and the money supply

Our formal question will be whether the velocity of money is a well defined concept, and whether it is useful in monetary control. To motivate both the question and the terms used in it, we begin with more operational questions:

1. To what extent is the money supply a well defined and effective control variable on economic activity?
2. How does this control function differ among economic sectors? Particularly, does it differ in the transmission of information, valuation and coordination in the various segments of the economy?
3. How does heterogeneity of the need and flow of money lead to fluctuations in the efficacy of control?

C. The Fisher equation and velocity

Economists, following Irving Fisher [35](p. 26), have attempted to formalize the control function of the money supply in a stylized equation $PQ = MV$, probably intended also to resemble a thermodynamic equation of state. Here P is supposed to represent some kind of aggregate price, Q the time rate of flow of goods traded at average price P, M the amount of money used in trade, and V its average time rate of turnover; that is, how many trades are mediated by each unit of money per unit time. V is (unfortunately) called the *velocity of money*, and has units of $(\text{time})^{-1}$.[8]

Whereas money is abstracted away in the general equilibrium analysis, and only allocation of physical resources matters, the Fisher equation intuitively captures the constraining capability of the money supply. In typical economic period-based models of trade, cash in advance requires that money be circulated at most once per period, meaning that the process description sets an upper limit on V of one turnover per period. Thus limitations in M imply limits in either the price or the quantity traded per period. Closely related to velocity and frequently measured in commercial practice is the number of inventory turns per annum, which is similarly utilized to gauge potential profitability in industries with thin selling margins.

Yet in much of macroeconomic thought it has been noted that the velocity of money may vary. Is this variation important? If, so why does it matter? The Fisher equation has served as the basis for the so-called "quantity theory of money". Among the factors influencing velocity that Fisher lists are: habit, technology, rate of interest and cyclical effects.

D. Deconstructing the Fisher equation

In reality, of course, there are many different combinations P_iQ_i, which may or may not have differently constrained characteristic V_i, and at the same time there are multiple money supplies M_j, which in different measures can mediate the various kinds of trade. The number of monies and near-monies is not constant through time and under changing technology, and the invention of new kinds of credit is one possible response to limitations in the existing money supply. Thus it is not at all clear whether the Fisher equation represents control by the money supply, or some much more complex optimization problem involving the simultaneous evolution and innovation of trade strategies, velocities, and money supplies.

Currently in the United States the money supply is defined at three levels, called M1, M2 and M3.

M1 has four components: (1) Currency[9], (2) Travelers checks, (3) deposits, and (4) other checkable deposits such as NOW accounts. Thus at least we need to consider four instruments in the aggregate description.

M2 includes M1 with the addition of: (1) money market funds, (2) retail savings deposits, (3) thrift institutions, (4) small-time deposits.

M3 includes M2 and: (1) large-time deposits, (2) repurchase agreements, (3) Eurodollars and (4) money market funds, for institutions only.

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game theory, presenting a higher level of abstraction than general equilibrium theory. It is explicitly normative in the sense that it assumes joint optimality as an expected property of any solution. Market games utilize the concept of totally balanced games [33, 34]. This approach enables us to explore the properties of a price system beyond the conventional feature of decentralized decision-making, and is covered in Ref. [28], Vol. III.

[8] A better appellation would use $\dot{Q}$ for the flow of goods, emphasizing that it is a rate, and make MV the "velocity of money", as its dimensions would then be an amount of money per unit time. However, the terminology is established, and to avoid further confusion we will follow convention.

[9] This include only currency outside of the Federal Reserve and Treasury.

	1980	1990	2000
GNP	2775	5832	9848
M1	408	824	1085
M2	1600	3277	4927
M3	1996	4152	7090

TABLE I: Gross national product (GNP) and money supplies M1, M2, and M3, for the United States in the years shown, in billions of dollars.

We note that the United States thus uses a 12-component definition of the money supply. A crude statistic on the orders of magnitude involved is indicate in Table I, which shows the three money supplies and gross national product for 1980, 1990 and 2000 measured in billions of dollars.

Even these crude numbers raise deep problems in conceptualization and measurement in economic theory and practice. In particular the turnover in the New York Stock Exchange in 2002 was 10492/9603 or over 109% of total valuation, and the NASDAQ was higher, yet the equilibrium models of economic theory can at best account for only a fraction of such trade, and at the high level of abstraction of general equilibrium theory the equilibrium may involve no transactions of shares at all.

The gap between financial reality and the economic theory of rational investor behavior is so large that a new emphasis on physically accurately defined process models, examined for both equilibrium and disequilibrium behavior, is called for. It must provide at least a conceptual framework and some crude estimates of how much of the actual trade could be accounted for by models invoking "rational expectations"(which is a euphemism for consistent stationary expectations), and how much would require some form of disequilibrium solution concept involving a yet-to-be-determined blend of behavioral factors involving growth, innovation and inductive behavior. At this point in the development of economics as a science we are far from having an adequate theory of economic dynamics, but with care it appears to be feasible to demonstrate the extreme limitations of equilibrium analysis in the financial markets. Doing so, however, requires not only the utilization of the methods of physics and game theory, but the recognition that the very essence of the financial system is that of a delicate fast acting multi-agent control mechanism over an economy in perpetual disequilibrium.

E. The money requirements of trade versus the money value of trade

The complexity of money supply as a control mechanism, and velocity as a mediating constraint, arises from the fact that the value of trade can have a different relation to the value *exchanged* in trade, in each sector of the economy and even for each good or service. Similarly, the cost of maintaining a money supply is quite different from the value-in-trade of the money itself. The heterogeneity papered over by Fisher's "aggregate" PQ may be disaggregated to some extent by a taxonomy of minimal models, capturing categories of constraint, and possibly providing more justification for aggregation within individual categories.

Some progress toward defining a natural money-measure of the value of trade has been made [8] for so-called "Gorman-aggregatable" economies [27, 32], with the most general measure being possible for economies whose agents have linear-separable utilities. These measures have furthermore been made a basis for endogenizing the problem of providing the money supply [25, 26], though only for commodity monies is there a fairly natural minimal model of the cost of a money supply in relation to its value-in-trade, created by the interest rate. All of these models have had simple, stationary time structure, and omitted production except through abstract endowment representations. An important and interesting problem that remains to be treated is how the interest rate is formed, which is essential to the costs of monies, but which governs very distinct processes of capital contribution to production, cognitive intertemporal discounting, and uncertainty.

IV. IS THERE EVER ENOUGH MONEY?

In the formal analysis of strategic market games the first emphasis has been on models that can be compared with the general equilibrium treatment of the same economic structure. If these models are solved utilizing equilibrium concepts, not only do they give (under the appropriate conditions) the allocations of primary goods that would have been obtained from general equilibrium theory, they also give insights concerning the money needed to cover transactions in an equilibrium state. The conditions for enough money in a timeless world are given below, but the world we live in is not timeless. In the disequilibrium world of innovation, mutation and failure there are local sources clamoring for resources and the purchase of these resources often calls for outside financing. This financing can be obtained in different ways from several sources. It can come from the government issue of money; the creation of near-money or bank debt from the banking system; the loan of existing money or near money by solvent financiers or other means. Whatever way is used, the financial system is acting as the perception and control system that is evaluating the economic worth of the call for funding and assigning and effectively rationing the funds among competing prospects. In this context the pressures are there for the creation of more funds.

It may even be the case that money supplies are adequate to enable complete trade for existing production and the consumption it enables, but that the problems of distribution and wealth accumulation cause cash flow constraints to limit trade anyway. Where luxury consumption is possible, there can be consistent forces away from money distributions that maximally serve total production in the society.

A. The concept of "enough money"

In casual discussion of the economics of production and exchange the topic of "does the economy have 'enough money' " is frequently asked. This concept can be made mathematically precise but surprisingly the answer to the question splits into two pieces, one of which is purely mathematical and the other heavily institutional, depending on the way in which the economy has been modeled as a set of rules designed to carry the contracting processes.

In essence the general equilibrium system with its focus on the wealth constraints on all individuals calls for the mathematics of the equations of wealth, whereas the strategic market games with their specification of cash flow constraints call for the mathematics of inequalities. Without going into the full technical detail, the heuristic argument is as follows. In a mass anonymous market a universally acceptable means of payment – a jointly recognizable and accepted money – may be utilized as a substitute for trust. However, if cash payments are required, unless credit or clearing arrangements are considered the cash payments impose cash-flow constraints on the optimization. In general these take the form of inequalities. The concept of "enough money well distributed" [28] is that none of these inequalities are binding on any agent in the economy. The analysis of enough money is completely well defined mathematically, but requires the full specification of the price formation mechanism and the attendant individual strategies.

B. Income distributions and the role of constraint

One empirical observation suggests, though, that the "enough money, well-distributed" of stationary strategic market games, with complete contracts and rational expectations, never applies to dynamical modern economies. That is the distribution of wealth and income in the economies of industrialized nations around the world [36]. It has been known since Pareto [37] that a significant fraction of the wealth in these economies is held by a small number of individuals (or families), whose density is a power-law in the wealth. Equivalently, the income accruing from ownership of wealth (as capital) is power-law distributed at large wealths, and may be exponentially or log-normally distributed [38, 39] for wage-earners. Though many examples of such distribution are known, they still have no conceptually and quantitatively adequate explanation. A remarkable feature of all these distributions is that they maximize the Shannon entropy [40] of the wealth or income distribution, subject to constraints [41, 42] on some function of wealth or income (its value, variance, or logarithm, for the various cases).

It may be a qualitatively robust result [38, 39] that the two distinct regions of wealth and income distribution are associated with additive and multiplicative randomization, suggesting a correspondence with wage versus capital dynamics. If this is so, there is persistent competition between the accumulation of wealth as capital, driven at least in part by innovation, and its distribution in the less flexible domain of wage labor. As long as such competition persists, the national economy by definition never has "enough money", and the shadow price of money from constraints on trade is always nonzero.

C. Steady-state economies?

In suggesting the universality of power-law distributions and wealth constraints, however, it is important to remember that all the economies studied have existed within the industrial age, a period of continual real earnings growth, roughly matched in physical terms by growth in per capita energy consumption [43, 44] (see also U. S. Department of Energy Report "Internal Energy Outlook 2005, Report # DOE/EIA-0484(2005), available at http://www.eia.doe.gov/oiaf/ieo). Some intriguing studies originating in biological allometric scaling [45] suggest that, despite the structural differences between economies and biological organisms, biological reproduction may respect universal constraints related to energy consumption, even when that energy is consumed industrially and mediated economically.

An interesting possibility is that economic growth is essentially an energetic phenomenon, in which case it cannot persist indefinitely on earth.[10] It may be that the competition between capital investment and wages would persist in a steady-state economy, as a result of a "red-queen effect" of persistent innovation [48]. Alternatively, it may be that the wealth constraints exhibited by industrial economies now are only reflections of innovation made possible by energetic growth, and would disappear in steady-state economies, leading to characteristic exponential wealth distributions associated with simple optimizing trade under uncertainty [49]. Predictions about the future of wealth distributions are thus good (and falsifiable) tests of our understanding of the dynamics and constraints of money and capital.

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[10] In recent times it has become increasingly common to call for economic attention to non-growth scenarios [46, 47].

V. INDEXING THE COMPLEXITY OF AN ECONOMY

A crude analogy can be made between the development of the one-celled organism to the progressively more complicated multicellular animal, and that of simple hunter-gatherer economies to the complex economies of today. New connections are established and new functions appear. In particular with the growth of the division of labor, of separate functions for many different types of corporation and partnerships, and the proliferation of anonymous mass markets, the need for coordination and evaluation has grown enormously. The standardization and utilization of many financial instruments over the last century has brought the role of finance to the fore as the control, organizing and perception device over the economy as a whole.

This process of innovation is clearly structured, and at some stages it is hierarchical. Markets and monies to some extent co-evolve, but the transition from commodity monies to government debt toward pure fiat monies has only occurred in the context of sophisticated markets, and what we would regard as modern frameworks for enforcing laws. Corporations and stocks and bonds are only required when large-scale production is combined with autonomy of capital ownership from the state, and successive layers of ownership paper and derivatives arise to disaggregate ever finer components of risk, as the cost of producing and monitoring such instruments comes down. The financial sector of a modern economy is invariably its highest-velocity sector, and the one most susceptible to fluctuations in usage patterns. Presumably this is because the value added by financial instruments comes primarily from information, while the cost of financial transactions is small. Small information value may be gained per transaction, but a small commitment of money suffices to cover a large transaction volume because of their high velocity. A certain technological sophistication is required to handle such large-volume, large-value-in-trade, small-value-added transactions at acceptable cost. We may thus ask whether, among the many measures of complexity posited [50], the position of an economy in the hierarchy of financial control is an index to its complexity.

A. The central role of risk

There has been an explosive development in the study of both the qualitative and quantitative properties of risk. The power of careful modeling and sophisticated stochastic analysis has already shown itself in the context of the stock market and other financial markets, but as the qualitative aspects of risk are being uncovered and well-defined, the scope of a useful econo-physics stretches far beyond the confines of the dynamics of paper on paper on the stockmarket, to starting to unravel the structures at the interface between where the neural system and broad control mechanism of the economy as a whole meets the physical entities over which (and whom) it has control. Conventional micro-economic theory has stressed production and consumption; applied macroeconomics has implicitly assumed a control role for government fiscal and monetary policies, but the change that is in the making is in the development of a general disequilibrium microeconomics with the intertwining of production, consumption and finance in ways that may even exceed the envisionings of writers such as Simmel [51], Schumpeter [52], von Mises [53], Hayek [54] or Keynes [55].

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