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Random graphs with arbitrary degree distribution and their applications

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Recent work on the structure of social networks and the internet has focussed attention on graphs with distributions of vertex degree that are significantly different from the Poisson degree distributions which have been widely studied in the past. In this paper we develop in detail the theory of random graphs with arbitrary degree distributions. In addition to simple undirected, unipartite graphs, we examine the properties of directed and bipartite graphs. Among other results, we derive exact expressions for the position of the phase transition at which a giant component first forms, the mean component size, the size of the giant component if there is one, the mean number of vertices a certain distance away from a randomly chosen vertex, and the average vertex–vertex distance within a graph. We also apply our theory to some real-world graphs, including the world-wide web and collaboration graphs of scientists and business-people. We demonstrate that in some cases random graphs with appropriate distributions of vertex degree predict with surprising accuracy the behavior of the real world, while in others there is a measurable discrepancy between theory and reality, perhaps indicating the presence of additional social structure in the network which is not captured by the random graph.

I. INTRODUCTION

A random graph [1] is a collection of points, or vertices, with lines, or edges, connecting pairs of them at random (Fig. 1a). The study of random graphs has a long history. Starting with the highly influential work of Paul Erdős and Albert Rényi in the 1950s and 1960s [2–4], random graph theory has developed into one of the mainstays of modern discrete mathematics, and has produced a prodigious number of results, many of them highly ingenious, describing statistical properties of graphs, such as distributions of component sizes, existence and size of a giant component, and typical vertex–vertex distances.

In almost all these studies the assumption has been made that the presence or absence of an edge between two vertices is independent of the presence or absence of any other edge, so that each edge may be considered to be present with independent probability $p$. If there are $N$ vertices in a graph, and each is connected to an average of $z$ edges, then it is trivial to show that $p = z/(N - 1)$, which for large $N$ is usually approximated by $z/N$. The number of edges connected to any particular vertex is called the degree $k$ of that vertex, and has a probability distribution $p_k$ given by

$$p_k = \binom{N}{k} p^k (1 - p)^{N-k} \simeq \frac{z^k e^{-z}}{k!},$$

where the second equality becomes exact in the limit of large $N$. This distribution we recognize as the Poisson distribution: the normal random graph has a Poisson distribution of vertex degrees, a point which turns out to be crucial, as we now explain.

![Random Graphs](https://example.com/random_graphs.png)

**FIG. 1.** (a) A schematic representation of a random graph, the circles representing vertices and the lines edges. (b) A directed random graph, i.e., one in which each edge runs in only one direction.

Random graphs however turn out to have severe shortcomings as models of real-world phenomena. Although it is difficult to determine experimentally the
structure of the network of contacts by which a disease is spread, studies have been performed of other social networks such as networks of friendships within a variety of communities [8–10], networks of telephone calls [11,12], airline timetables [13], and the power grid [14], as well as networks in physical or biological systems, including neural networks [14], the structure and conformation space of polymers [15,16], and metabolic pathways [17]. It is found [12,13] that the distribution of vertex degrees in many of these networks is measurably different from a Poisson distribution—often wildly different—and this strongly suggests, as has been emphasized elsewhere [18], that there are features of such networks which we would miss if we were to approximate them by a normal (Poisson) random graph.

Another very widely studied network is the internet, whose structure has attracted an exceptional amount of scrutiny, academic and otherwise, following its meteoric rise to public visibility starting in 1993. Pages on the world-wide web may be thought of as the vertices of a graph and the hyperlinks between them as edges. Empirical studies [19–21] have shown show that this graph has a distribution of vertex degree which is heavily right-skewed and possesses a fat (power-law) tail with an exponent between $-2$ and $-3$. (The underlying physical structure of the internet also has a degree distribution of this type [22].) This distribution is very far from Poisson, and therefore we would expect that a simple random graph would give a very poor approximation of the structural properties of the web. However, the web differs from a random graph in another way also: it is directed. Links on the web lead from one page to another in only one direction (see Fig. 1b). As discussed by Broder et al. [21] this has a significant practical effect on the typical accessibility of one page from another, and this effect also will not be captured by a simple (undirected) random graph model.

A further class of networks that has attracted scrutiny is the class of collaboration networks. Examples of such networks include the boards of directors of companies [23,24], co-ownership networks of companies [25], and collaborations of scientists [26] and movie actors [14]. As well as having strongly non-Poisson degree distributions [13,26], these networks have a bipartite structure; there are two distinct kinds of vertices on the graph with links running only between vertices of unlike kinds [27,28]—see Fig. 2. In the case of movie actors, for example, the two types of vertices are movies and actors, and the network can be represented as a graph with edges running between each movie and the actors that appear in it. Researchers have also considered the projection of this graph onto the unipartite space of actors only, also called a one-mode network. In such a projection two actors are considered connected if they have appeared in a movie together. The construction of the one-mode network however involves discarding some of the information contained in the original bipartite network, and for this reason it is more desirable to model collaboration networks using the full bipartite structure.

Given the high current level of interest in the structure of many of the graphs described here, and given their substantial differences from the ordinary random graphs that have been studied in the past, it would clearly be useful if we could generalize the mathematics of random graphs to non-Poisson degree distributions, and to directed and bipartite graphs. In this paper we do just that, demonstrating in detail how the statistical properties of each of these graph types can be calculated exactly in the limit of large graph size. We also give examples of the application of our theory to the modeling of a number of real-world networks, including the world-wide web and collaboration graphs. A brief account of some of the work reported here has appeared previously as Ref. [28].

II. RANDOM GRAPHS OF ARBITRARY DEGREE DISTRIBUTION

In this section we develop a formalism for calculating a variety of quantities, both local and global, on large unipartite undirected graphs with arbitrary probability distribution of the degrees of their vertices. In all respects other than their degree distribution, these graphs are assumed to be entirely random. By this we mean that all properties calculated are averaged uniformly over the ensemble of graphs with the desired degree distribution. Some results are already known for graphs of this type: in two beautiful recent papers, Molloy and Reed [29,30] have derived formulas for the position of the phase transition at which a giant component first appears, and the
size of the giant component. Our formalism yields an alternative derivation of these results and also provides a framework for obtaining other quantities of interest, some of which we calculate here. In Sections III and IV we extend our formalism to the case of directed graphs (such as the world-wide web) and bipartite graphs (such as collaboration graphs).

A. Generating functions

Our approach is based on generating functions [31], the most fundamental of which, for our purposes, is the generating function \( G_0(x) \) for the probability distribution of vertex degrees \( k \). Suppose that we have a unipartite undirected graph—an acquaintance network, for example—of \( N \) vertices, with \( N \) large. We define

\[
G_0(x) = \sum_{k=0}^{\infty} p_k x^k,
\]

where \( p_k \) is the probability that a randomly chosen vertex on the graph has degree \( k \). The distribution \( p_k \) is assumed correctly normalized, so that

\[
G_0(1) = 1.
\]

The same will be true of all generating functions considered here, with a few important exceptions, which we will note at the appropriate point. The normalization of the probability distribution also implies that \( G_0(x) \) is finite for all \(|x| \leq 1\), although it may have poles outside this region. All the calculations of this paper will be confined to the region \(|x| \leq 1\).

The function \( G_0(x) \), and indeed any probability generating function, has a number of properties that will prove useful in subsequent developments.

Derivatives The probability \( p_k \) is given by the \( k^{th} \) derivative of \( G_0 \) according to

\[
p_k = \left. \frac{1}{k!} \frac{d^k G_0}{dx^k} \right|_{x=0}.
\]

Thus the one function \( G_0(x) \) encapsulates all the information contained in the discrete probability distribution \( p_k \). We say that the function \( G_0(x) \) “generates” the probability distribution \( p_k \).

Moments The average over the probability distribution generated by a generating function—for instance, the average degree \( z \) of a vertex in the case of \( G_0(x) \)—is given by

\[
z = \langle k \rangle = \sum_k k p_k = G_0'(1).
\]

Thus if we can calculate a generating function we can also calculate the mean of the probability distribution which it generates. Higher moments of the distribution can be calculated from higher derivatives also. In general, we have

\[
\langle k^n \rangle = \sum_k k^n p_k = \left[ \left( x \frac{d}{dx} \right)^n G_0(x) \right]_{x=1}.
\]

Powers If the distribution of a property \( k \) of an object is generated by a given generating function, then the distribution of the total of \( k \) summed over \( m \) independent realizations of the object is generated by the \( m^{th} \) power of that generating function. For example, if we choose \( m \) vertices at random from a large graph, then the distribution of the sum of the degrees of those vertices is generated by \( [G_0(x)]^m \). To see why this is so, consider the simple case of just two vertices. The square \( [G_0(x)]^2 \) of the generating function for a single vertex can be expanded as

\[
\left[ \sum_k p_k x^k \right]^2 = \sum_{jk} p_j p_k x^{j+k}
\]

\[
= p_0 p_0 x^0 + (p_0 p_1 + p_1 p_0) x^1
\]

\[
+ (p_0 p_2 + p_1 p_1 + p_2 p_0) x^2
\]

\[
+ (p_0 p_3 + p_1 p_2 + p_2 p_1 + p_3 p_0) x^3 + \ldots
\]

It is clear that the coefficient of the power of \( x^n \) in this expression is precisely the sum of all products \( p_j p_k \) such that \( j + k = n \), and hence correctly gives the probability that the sum of the degrees of the two vertices will be \( n \). It is straightforward to convince oneself that this property extends also to all higher powers of the generating function.

All of these properties will be used in the derivations given in this paper.

Another quantity that will be important to us is the distribution of the degree of the vertices which we arrive at by following a randomly chosen edge. Such an edge arrives at a vertex with probability proportional to the degree of that vertex, and the vertex therefore has a probability distribution of degree proportional to \( k p_k \). The correctly normalized distribution is therefore generated by

\[
\frac{\sum_k k p_k x^k}{\sum_k k p_k} = x \frac{G_0'(x)}{G_0(1)}.
\]

If we start at a randomly chosen vertex and follow each of the edges at that vertex to reach the \( k \) nearest neighbors, then the vertices arrived at each have the distribution of remaining outgoing edges generated by this function, less one power of \( x \), to allow for the edge that we arrived along. Thus the distribution of outgoing edges is generated by the function

\[
G_1(x) = \frac{G_0'(x)}{G_0(1)} = \frac{1}{z} G_0'(x),
\]
where $z$ is the average vertex degree, as before. The probability that any of these outgoing edges connects to the original vertex that we started at, or to any of its other immediate neighbors, goes as $N^{-1}$ and hence can be neglected in the limit of large $N$. Thus, making use of the “powers” property of the generating function described above, the generating function for probability distribution of the number of second neighbors of the original vertex can be written as

\[ \sum_k p_k[G_1(x)]^k = G_0(G_1(x)). \] (10)

Similarly, the distribution of third-nearest neighbors is generated by $G_0(G_1(G_1(x)))$, and so on. The average number $z_2$ of second neighbors is

\[ z_2 = \left[ \frac{d}{dx} G_0(G_1(x)) \right]_{x=1} = G_0'(1)G_1'(1), \] (11)

where we have made use of the fact that $G_1(1) = 1$.

### B. Examples

To make things more concrete, we immediately introduce some examples of specific graphs to illustrate how these calculations are carried out.

**a. Poisson-distributed graphs** The simplest example of a graph of this type is one for which the distribution of degree is binomial, or Poisson in the large $N$ limit. This distribution yields the standard random graph studied by many mathematicians and discussed in Section I. In this graph the probability $p = z/N$ of the existence of an edge between any two vertices is the same for all vertices, and $G_0(x)$ is given by

\[ G_0(x) = \sum_{k=0}^{N} \binom{N}{k} p^k(1-p)^{N-k} x^k = (1 - p + px)^N = e^{z(x-1)}, \] (12)

where the last equality applies in the limit $N \to \infty$. It is now trivial to show that the average degree of a vertex is indeed $G_0'(1) = z$, as we would expect, and that the probability distribution of degree is given by

\[ p_k = \frac{z^k e^{-z}}{k!}, \] (13)

which is the normal Poisson distribution. Notice also that for this special case we have $G_1(x) = G_0(x)$, so that the distribution of outgoing edges at a vertex is the same, regardless of whether we arrived there by choosing a vertex at random, or by following a randomly chosen edge. This property, which is peculiar to the Poisson-distributed random graph, is the reason why the theory of random graphs of this type is especially simple.

**b. Exponentially distributed graphs** Perhaps the next simplest type of graph is one with an exponential distribution of vertex degrees

\[ p_k = (1 - e^{-1/\kappa}) e^{-k/\kappa}, \] (14)

where $\kappa$ is a constant. The generating function for this distribution is

\[ G_0(x) = (1 - e^{-1/\kappa}) \sum_{k=0}^{\infty} e^{-k/\kappa} x^k = \frac{1 - e^{-1/\kappa}}{1 - xe^{-1/\kappa}}, \] (15)

and

\[ G_1(x) = \left[ \frac{1 - e^{-1/\kappa}}{1 - xe^{-1/\kappa}} \right]^2. \] (16)

An example of a graph with an exponential degree distribution is given in Section V A.

**c. Power-law distributed graphs** The recent interest in the properties of the world-wide web and of social networks leads us to investigate the properties of graphs with a power-law distribution of vertex degrees. Such graphs have been discussed previously by Barabási et al. [18,19] and by Aiello et al. [12]. In this paper, we will look at graphs with degree distribution given by

\[ p_k = C k^{-\tau} e^{-k/\kappa} \quad \text{for } k \geq 1. \] (17)

where $C$, $\tau$, and $\kappa$ are constants. The reason for including the exponential cutoff is two-fold: first many real-world graphs appear to show this cutoff [13,26]; second it makes the distribution normalizable for all $\tau$, and not just $\tau \geq 2$.

The constant $C$ is fixed by the requirement of normalization, which gives $C = \left[ \text{Li}_n(e^{-1/\kappa}) \right]^{-1}$ and hence

\[ p_k = \frac{k^{-\tau} e^{-k/\kappa}}{\text{Li}_n(e^{-1/\kappa})} \quad \text{for } k \geq 1, \] (18)

where $\text{Li}_n(x)$ is the $n^{th}$ polylogarithm of $x$, a function familiar to those who have worked with Feynman integrals.

Substituting (18) into Eq. (2), we find that the generating function for graphs with this degree distribution is

\[ G_0(x) = \frac{\text{Li}_n(\kappa e^{-1/\kappa})}{\kappa e^{-1/\kappa}}, \] (19)

In the limit $\kappa \to \infty$—the case considered in Refs. [19] and [12]—this simplifies to

\[ G_0(x) = \frac{\text{Li}_n(x)}{\zeta(\tau)}, \] (20)

where $\zeta(x)$ is the Riemann $\zeta$-function. The function $G_1(x)$ is given by

\[ G_1(x) = \frac{\text{Li}_{n-1}(\kappa e^{-1/\kappa})}{\kappa e^{-1/\kappa}}. \] (21)
Thus, for example, the average number of neighbors of a randomly-chosen vertex is
\[
z = G'_0(1) = \frac{\text{Li}_{r-1}(e^{-1/\kappa})}{\text{Li}_r(e^{-1/\kappa})}, \tag{22}
\]
and the average number of second neighbors is
\[
z_2 = G'_0(1)G'_1(1) = \frac{\text{Li}_{r-2}(e^{-1/\kappa}) - \text{Li}_{r-1}(e^{-1/\kappa})}{\text{Li}_r(e^{-1/\kappa})}. \tag{23}
\]

d. Graphs with arbitrary specified degree distribution. In some cases we wish to model specific real-world graphs which have known degree distributions—known because we can measure them directly. A number of the graphs described in the introduction fall into this category. For these graphs, we know the exact numbers \( n_k \) of vertices having degree \( k \), and hence we can write down the exact generating function for that probability distribution in the form of a finite polynomial:
\[
G_0(x) = \frac{\sum_k n_k x^k}{\sum_k n_k}, \tag{24}
\]
where the sum in the denominator ensures that the generating function is properly normalized. As an example, suppose that in a community of 1000 people, each person knows between zero and five of the others, the exact numbers of people in each category being, from zero to five: \(86, 150, 363, 238, 109, 54\). This distribution will then be generated by the polynomial
\[
G_0(x) = \frac{86 + 150x + 363x^2 + 238x^3 + 109x^4 + 54x^5}{1000}. \tag{25}
\]

C. Component sizes

We are now in a position to calculate some properties of interest for our graphs. First let us consider the distribution of the sizes of connected components in the graph. Let \( H_1(x) \) be the generating function for the distribution of the sizes of components which are reached by choosing a random edge and following it to one of its ends. The component one finds by doing this can be represented graphically as in Fig. 3, and hence, making use of the “powers” property of Section II A, \( H_1(x) \) satisfies the iterative equation
\[
H_1(x) = xG_1(H_1(x)). \tag{26}
\]
If we start at a randomly chosen vertex, then we have one such component at the end of each edge leaving that vertex, and hence the generating function for the size of the whole component is
\[
H_0(x) = xG_0(H_1(x)). \tag{27}
\]

In principle, therefore, given the functions \( G_0(x) \) and \( G_1(x) \), we can solve Eq. (26) for \( H_1(x) \) and substitute into Eq. (27) to get \( H_0(x) \). Then we can find the probability that a randomly chosen vertex belongs to a component of size \( s \) by taking the \( s \)th derivative of \( H_0 \). In practice, unfortunately, this is usually impossible; Eq. (26) is a complicated and frequently transcendental equation, which rarely has a known solution. On the other hand, we note that the order \( x^s \) coefficient in the Taylor expansion of \( H_1(x) \) (and therefore also the \( s \)th derivative) are given exactly by only \( s+1 \) iterations of Eq. (27), starting with \( H_1 = 1 \), so that the distribution generated by \( H_0(x) \) can be calculated exactly to finite order in finite time. With current symbolic manipulation programs, it is quite possible to evaluate the first one hundred or so derivatives in this way. Failing this, an approximate solution can be found by numerical iteration and the distribution of cluster sizes calculated from Eq. (4) by numerical differentiation. Since direct evaluation of numerical derivatives is prone to machine-precision problems, we recommend evaluating the derivatives by numerical integration of the Cauchy formula, giving the probability distribution \( P_s \) of cluster sizes thus:
\[
P_s = \frac{1}{s!} \frac{d^sH_0}{dz^s} \bigg|_{z=0} = \frac{1}{2\pi i} \oint \frac{H_0(z)}{z^{s+1}}dz. \tag{28}
\]
The best numerical precision is obtained by using the largest possible contour, subject to the condition that it enclose no poles of the generating function. The largest contour for which this condition is satisfied in general is the unit circle \( |z| = 1 \) (see Section II A), and we recommend using this contour for Eq. (28). It is possible to find the first thousand derivatives of a function without difficulty using this method [32].

D. Mean component size, the phase transition, and the giant component

Although it is not usually possible to find a closed-form expression for the complete distribution of cluster sizes...
on a graph, we can find closed-form expressions for the average properties of clusters from Eqs. (26) and (27). For example, the average component size is given in the normal fashion by

\[ \langle s \rangle = H'_0(1) = 1 + G'_0(1)H'_1(1). \]  

(29)

From Eq. (26) we have

\[ H'_1(1) = 1 + G'_1(1)H'_1(1), \]  

(30)

and hence

\[ \langle s \rangle = 1 + \frac{G'_0(1)}{1 - G'_1(1)}. \]  

(31)

We see that this expression diverges when

\[ G'_1(1) = 1, \]  

(32)
or equivalently when

\[ G''_0(1) = G'_0(1). \]  

(33)

This point marks the phase transition at which a giant component first appears. Substituting Eq. (2) into Eq. (33), we can also write the condition for the phase transition as

\[ \sum_k k(k - 2)p_k = 0. \]  

(34)

Indeed, since this sum increases monotonically as edges are added to the graph, it follows that the giant component exists when this sum is positive, and does not exist when it is negative. This result has been derived by different means by Molloy and Reed [29].

Our generating function formalism still works when there is a giant component in the graph, but \( H_0(x) \) then generates the probability distribution of the sizes of components excluding the giant component [32]. This means that \( H_0(1) \) is no longer unity, as it is for the other generating functions considered so far, but instead takes the value 1 – \( S \), where \( S \) is the fraction of the graph occupied by the giant component. We can use this to calculate the size of the giant component from Eqs. (26) and (27) thus:

\[ S = 1 - G_0(u), \]  

(35)

where \( u \) is the smallest non-negative real solution of

\[ u = G_1(u). \]  

(36)

This result has been derived in a different but equivalent form by Molloy and Reed [30], using different methods.

For example, in the ordinary random graph with Poisson degree distribution, we have \( G_0(x) = G_1(x) = e^{x(1-x)} \) (Eq. (12)), and hence we find simply that \( 1 - S = u \) is a solution of \( u = G_0(u) \), or equivalently that

\[ S = 1 - e^{-u}, \]  

(37)

which is a well-known result [1]. For graphs with purely power-law distributions (\( \kappa \to \infty \)), \( S \) is given by (35) with \( u \) the smallest non-negative real solution of

\[ u^2 = \frac{\text{Li}_{-1}(u)}{\zeta(\tau - 1)}. \]  

(38)

For all \( \tau \leq 2 \) this gives \( u = 0 \), and hence \( S = 1 \), implying that a randomly chosen vertex belongs to the giant component with probability tending to 1 as \( \kappa \to \infty \). For graphs with \( \tau > 2 \), this is never the case, even for infinite \( \kappa \). These results have been derived by different means by Aiello et al. [12].

E. Asymptotic form of the cluster size distribution

A variety of results are known about the asymptotic properties of the coefficients of generating functions, some of which can usefully be applied to the distribution of cluster sizes generated by \( H_0(x) \). Close to the phase transition, we expect the tail of the distribution \( P_s \) to behave as

\[ P_s \sim s^{-\alpha}e^{-s/s^*}, \]  

(39)

where the constants \( \alpha \) and \( s^* \) can be calculated from the properties of \( H_0(x) \) as follows.

The cutoff parameter \( s^* \) is simply related to the radius of convergence \( |x^*| \) of the generating function [31,33], according to

\[ s^* = \frac{1}{\log |x^*|}. \]  

(40)

The radius of convergence \( |x^*| \) is equal to the magnitude of the position \( x^* \) of the singularity in \( H_0(x) \) nearest to the origin. From Eq. (27) we see that such a singularity may arise either through a singularity in \( G_0(x) \) or through one in \( H_1(x) \). However, since the first singularity in \( G_0(x) \) is known to be outside the unit circle (Section II A), and the first singularity in \( H_1(x) \) tends to \( x = 1 \) as we go to the phase transition (see below), it follows that, sufficiently close to the phase transition, the singularity in \( H_0(x) \) closest to the origin is also a singularity in \( H_1(x) \). With this result \( x^* \) is easily calculated.

Although we do not in general have a closed-form expression for \( H_1(x) \), it is easy to derive one for its functional inverse. Putting \( w = H_1(x) \) and \( x = H_1^{-1}(w) \) in Eq. (26) and rearranging, we find

\[ x = H_1^{-1}(w) = \frac{w}{G_1(w)}. \]  

(41)

The singularity of interest corresponds to the point \( w^* \) at which the derivative of \( H_1^{-1}(w) \) is zero, which is a solution of

\[ G_1(w^*) - w^*G'_1(w^*) = 0. \]  

(42)
Then \( x^* \) (and hence \( s^* \)) is given by Eq. (41). Note that there is no guarantee that (42) has a finite solution, and that if it does not, then \( P_s \) will not in general follow the form of Eq. (39).

When we are precisely at the phase transition of our system, we have \( G_1'(1) = G_1(1) = 1 \), and hence the solution of Eq. (42) gives \( w^* = x^* = 1 \)—a result which we used above—and \( s^* \to \infty \). We can use the fact that \( x^* = 1 \) at the transition to calculate the value of the exponent \( \alpha \) as follows. Expanding \( H_1^{-1}(w) \) about \( w^* = 1 \) by putting \( w = 1 + \epsilon \) in Eq. (41), we find that

\[
H_1^{-1}(1 + \epsilon) = 1 - \frac{1}{2} G_1''(1) \epsilon^2 + O(\epsilon^3),
\]

where we have made use of \( G_1(1) = G_1'(1) = 1 \) at the phase transition. So long as \( G_1''(1) \neq 0 \), which in general it is not, this implies that \( H_1(x) \) and hence also \( H_0(x) \) are of the form

\[
H_0(x) \sim (1 - x)^\beta \quad \text{as} \quad x \to 1,
\]

with \( \beta = \frac{1}{2} \). This exponent is related to the exponent \( \alpha \) as follows. Equation (39) implies that \( H_0(x) \) can be written in the form

\[
H_0(x) = \sum_{s=0}^{\alpha-1} P_s x^s + C \sum_{s=\alpha}^{\infty} s^{-\alpha} e^{-s/x} x^s + \epsilon(a),
\]

where \( C \) is a constant and the last (error) term \( \epsilon(a) \) is assumed much smaller than the second term. The first term in this expression is a finite polynomial and therefore has no singularities on the finite plane; the singularity resides in the second term. Using this equation, the exponent \( \beta \) can be written:

\[
\beta = \lim_{x \to 1} \left[ 1 + (x - 1) \frac{H_0''(x)}{H_0'(x)} \right] = \lim_{x \to 1} \left[ \frac{1}{x} \frac{1 - x \sum_{s=0}^{\alpha-1} s^{-\alpha} x^s - 1}{\sum_{s=\alpha}^{\infty} s^{-\alpha} x^s - 1} \right] = \lim_{x \to 1} \left[ \frac{1}{x} \frac{1 - x \sum_{s=\alpha}^{\infty} s^{-\alpha} x^s - 1}{\sum_{s=0}^{\alpha-1} s^{-\alpha} x^s - 1} \right],
\]

where we replace the sums with integrals when \( a \) is large, and \( \Gamma(\nu, \mu) \) is the incomplete \( \Gamma \)-function. Taking the limits in the order specified and rearranging for \( \alpha \), we then get

\[
\alpha = \beta + 1 = \frac{4}{3},
\]

regardless of degree distribution, except in the special case where \( G_1''(1) \) vanishes (see Eq. (43)). The result \( \alpha = \frac{4}{3} \) was known previously for the ordinary Poisson random graph [1], but not for other degree distributions.

**F. Numbers of neighbors and average path length**

We turn now to the calculation of the number of neighbors who are \( m \) steps away from a randomly chosen vertex. As shown in Section II A, the probability distributions for first- and second-nearest neighbors are generated by the functions \( G_0(x) \) and \( G_0(G_1(x)) \). By extension, the distribution of \( m \)th neighbors is generated by \( G_0(G_1(\ldots G_1(x) \ldots)) \), with \( m - 1 \) iterations of the function \( G_1 \) acting on itself. If we define \( G(m)(x) \) to be this generating function for \( m \)th neighbors, then we have

\[
G(m)(x) = \begin{cases} 
G_0(x) & \text{for } m = 1 \\
G(m-1)(G_1(x)) & \text{for } m \geq 2.
\end{cases}
\]

Then the average number \( z_m \) of \( m \)th-nearest neighbors is

\[
z_m = \frac{\mathrm{d}G(m)}{\mathrm{d}x} \bigg|_{x=1} = G_1'(1)G(m-1)'(1) = G_1'(1)z_{m-1}.
\]

Along with the initial condition \( z_1 = z = G_0'(1) \), this then tells us that

\[
z_m = [G_1'(1)]^{m-1} G_0'(1) \left[ \frac{z_2}{z_1} \right]^{m-1} z_1.
\]

From this result we can make an estimate of the typical length \( \ell \) of the shortest path between two randomly chosen vertices on the graph. This typical path length is reached approximately when the total number of neighbors of a vertex out to that distance is equal to the number of vertices on the graph, i.e., when

\[
1 + \sum_{m=1}^{\ell} z_m = N.
\]

Using Eq. (50) this gives us

\[
\ell = \log((N - 1)[z_2 - z_1] + z_2^2) - \log z_2^2 \frac{\log(z_2/z_1)}{\log(z_2/z_1) + 1}.
\]

In the common case where \( N \gg z_1 \) and \( z_2 \gg z_1 \), this reduces to

\[
\ell = \frac{\log(N/z_1)}{\log(z_2/z_1)} + 1.
\]

This result is only approximate for two reasons. First, the conditions used to derive it are only an approximation; the exact answer depends on the detailed structure of the graph. Second, it assumes that all vertices are reachable from a randomly chosen starting vertex. In general however this will not be true. For graphs with no giant component it is certainly not true and Eq. (53) is meaningless. Even when there is a giant component however, it is usually not the case that it fills the entire graph. A better approximation to \( \ell \) may therefore be given by replacing \( N \) in Eq. (53) by \( NS \), where \( S \) is the fraction of the graph occupied by the giant component, as in Section II D.

Such shortcomings notwithstanding, there are a number of remarkable features of Eq. (53):...
1. It shows that the average vertex–vertex distance for all random graphs, regardless of degree distribution, should scale logarithmically with size $N$, according to $\ell = A + B \log N$, where $A$ and $B$ are constants. This result is of course well-known for a number of special cases.

2. It shows that the average distance, which is a global property, can be calculated from a knowledge only of the average numbers of first- and second-nearest neighbors, which are local properties. It would be possible therefore to measure these numbers empirically by purely local measurements on a graph such as an acquaintance network and from them to determine the expected average distance between vertices.

3. It shows that only the average numbers of first- and second-nearest neighbors are important to the calculation of average distances, and thus that two graphs with completely different distributions of vertex degrees, but the same values of $z_1$ and $z_2$, will have the same average distances.

For the case of the purely theoretical example graphs we discussed earlier, we cannot make an empirical measurement of $z_1$ and $z_2$, but we can still employ Eq. (53) to calculate $\ell$. In the case of the normal (Poisson) random graph, for instance, we find from Eq. (12) that $z_1 = z$, $z_2 = z^2$, and so $\ell = \log N / \log z$, which is the standard result for graphs of this type [1].

For the graph with power-law distributed degree according to Eq. (18), $z_1$ and $z_2$ are given by Eqs. (22) and (23), and the average vertex–vertex distance is

$$\ell = \frac{\log N + \log \left[ \text{Li}_2\left(e^{-1/\kappa}\right)/\text{Li}_{\tau-1}\left(e^{-1/\kappa}\right)\right]}{\log \left[ \text{Li}_{\tau-2}\left(e^{-1/\kappa}\right)/\text{Li}_{\tau-1}\left(e^{-1/\kappa}\right)\right]} + 1.$$  \hspace{1cm} (54)

In the limit $\kappa \to \infty$, this becomes

$$\ell = \frac{\log N + \log \left[ \zeta(\tau)/\zeta(\tau - 1)\right]}{\log \left[ \zeta(\tau - 2)/\zeta(\tau - 1)\right]} + 1.$$  \hspace{1cm} (55)

Note that this expression does not have a finite positive real value for any $\tau < 3$, indicating that one must specify a finite cutoff $\kappa$ for the degree distribution to get a well-defined average vertex–vertex distance on such graphs.

**G. Simulation results**

As a check on the results of this section, we have performed extensive computer simulations of random graphs with various distributions of vertex degree. Such graphs are relatively straightforward to generate. First, we generate a set of $N$ random numbers $\{k_i\}$ to represent the degrees of the $N$ vertices in the graph. These may be thought of as the “stubs” of edges, emerging from their respective vertices. Then we choose pairs of these stubs at random and place edges on the graph joining them up. It is simple to see that this will generate all graphs with the given set of vertex degrees with equal probability. The only small catch is that the sum $\sum_i k_i$ of the degrees must be even, since each edge added to the graph must have two ends. This is not difficult to contrive however. If the set $\{k_i\}$ is such that the sum is odd, we simply throw it away and generate a new set.

As a practical matter, integers representing vertex degrees with any desired probability distribution can be generated using the transformation method if applicable, or failing that, a rejection or hybrid method [34]. For example, degrees obeying the power-law-plus-cutoff form of Eq. (18) can be generated using a two-step hybrid transformation/rejection method as follows. First, we generate random integers $k \geq 1$ with distribution proportional to $e^{-k/\kappa}$ using the transformation

$$k = \left[ -\kappa \log(1 - r) \right],$$  \hspace{1cm} (56)

where $r$ is a random real number uniformly distributed in the range $0 \leq r < 1$. Second, we accept this number with probability $k^{-\tau}$, where by “accept” we mean that if the number is not accepted we discard it and generate another one according to Eq. (56), repeating the process until one is accepted.

In Fig. 4 we show results for the size of the giant component in simulations of undirected unipartite graphs with vertex degrees distributed according to Eq. (18) for a variety of different values of $\tau$ and $\kappa$. On the same plot
we also show the expected value of the same quantity derived by numerical solution of Eqs. (35) and (36). As the figure shows, the agreement between simulation and theory is excellent.

**III. DIRECTED GRAPHS**

We turn now to directed graphs with arbitrary degree distribution. An example of such a graph is the worldwide web, which is directed since every hyperlink between two pages goes in only one direction, and which has a degree distribution which follows a power-law, as discussed in Section I.

Directed graphs introduce a subtlety that is not present in undirected ones, which becomes important when we apply our generating function formalism. In a directed graph it is not possible to talk about a “component”—i.e., a group of connected vertices—because even if vertex A can be reached by following (directed) edges from vertex B, that does not necessarily mean that vertex B can be reached from vertex A. There are in fact two correct generalizations of the idea of the component to a directed graph: the set of vertices which are reachable from a given vertex, and the set from which a given vertex can be reached. We will refer to these as “out-components” and “in-components” respectively. An in-component can also be thought of as those vertices reachable by following edges backwards (but not forwards) from a specified vertex. It is possible to study directed graphs by allowing both forward and backward traversal of edges (see Ref. [21], for example). In this case, however, the graph effectively becomes undirected and should be treated with the formalism of Section II.

With these considerations in mind, we now develop the generating function formalism appropriate to random directed graphs with arbitrary degree distribution.

**A. Generating functions**

In a directed graph, each vertex has separate in-degree and out-degree for links running into and out of that vertex. Let us define \( p_{jk} \) to be the probability that a randomly chosen vertex has in-degree \( j \) and out-degree \( k \). It is important to realize that in general this joint distribution of \( j \) and \( k \) is not equal to the product \( p_j p_k \) of the separate distributions of in- and out-degree. In the world-wide web, for example, it seems likely (although this question has not been investigated to our knowledge) that sites with a large number of out-going links also have a large number of in-going ones, i.e., that \( j \) and \( k \) are correlated, so that \( p_{jk} \neq p_j p_k \). We would like to appeal to those working on studies of the structure of the web to measure the joint distribution of in- and out-degrees of sites; empirical data on this distribution would make theoretical work much easier!

We now define a generating function for the joint probability distribution of in- and out-degrees, which is necessarily a function of two independent variables, \( x \) and \( y \), thus:

\[
G(x, y) = \sum_{jk} p_{jk} x^j y^k.
\]  

(57)

Since every edge on a directed graph must leave some vertex and enter another, the net average number of edges entering a vertex is zero, and hence \( p_{jk} \) must satisfy the constraint

\[
\sum_{jk} (j - k)p_{jk} = 0.
\]  

(58)

This implies that \( G(x, y) \) must satisfy

\[
\frac{\partial G}{\partial x} \bigg|_{x,y=1} = \frac{\partial G}{\partial y} \bigg|_{x,y=1} = z,
\]  

(59)

where \( z \) is the average degree (both in and out) of vertices in the graph.

Using the function \( G(x, y) \), we can, as before, define generating functions \( G_0 \) and \( G_1 \) for the number of outgoing edges leaving a randomly chosen vertex, and the number leaving the vertex reached by following a randomly chosen edge. We can also define generating functions \( F_0 \) and \( F_1 \) for the number arriving at such a vertex. These functions are given by

\[
F_0(x) = G(x, 1), \quad F_1(x) = \frac{1}{z} \frac{\partial G}{\partial y} \bigg|_{y=1},
\]  

(60)

\[
G_0(y) = G(1, y), \quad G_1(y) = \frac{1}{z} \frac{\partial G}{\partial x} \bigg|_{x=1}.
\]  

(61)

Once we have these functions, many results follow as before. The average numbers of first and second neighbors reachable from a randomly chosen vertex are given by

\[
z_1 = G_0(1), \quad z_2 = G_0(1)G_1(1),
\]  

(62)

with similar expressions involving \( F_0(x) \) and \( F_1(x) \) for numbers of vertices from which the chosen vertex can be reached. We can also make an estimate of the average path length on the graph from

\[
\ell = \frac{\log(N/z_1)}{\log(z_2/z_1)} + 1,
\]  

(63)

as before. However, this equation should be used with caution. As discussed in Section II.F, the derivation of this formula assumes that we are in a regime in which the bulk of the graph is reachable from most vertices. On a directed graph however, this may be far from true, as appears to be the case with the world-wide web [21]. One could estimate the diameter of the web by measuring the average numbers of first and second neighbors reachable from sites and plugging them into Eq. (63), but it is
not clear that this would give more than a rudimentary order-of-magnitude estimate.

The probability distribution of the numbers of vertices reachable from a randomly chosen vertex in a directed graph—i.e., of the sizes of the out-components—is generated by the function $H_0(y) = yG_0(H_1(y))$, where $H_1(y)$ is a solution of $H_1(y) = yG_1(H_1(y))$, just as before. (A similar and obvious pair of equations governs the sizes of the in-components.) The results for the asymptotic behavior of the component size distribution from Section II E generalize straightforwardly to directed graphs.

The average out-component size (not including the giant component, if there is one) is given by Eq. (31), and thus the point at which a giant component first appears is given once more by $G'_1(1) = 1$, or equivalently by

$$\frac{\partial^2 G}{\partial x \partial y} \bigg|_{x,y=1} = z. \quad (64)$$

Substituting Eq. (57) into this expression and using Eq. (59) gives the explicit condition

$$\sum_{jk} (2jk - j - k)p_{jk} = 0 \quad (65)$$

for the first appearance of the giant component. This expression is the equivalent of Eq. (34) for the directed graph. It is also possible, and equally valid, to define the position at which the giant component appears by $F'_1(1) = 1$, which provides an alternative derivation for the conditions (64) and (65).

Just as with the individual in- and out-components for vertices, the size of the giant component on a directed graph can also be defined in different ways. The giant component can be represented using the “bow-tie” diagram of Broder et al. [21], which we depict (in simplified form) in Fig. 5. The diagram has three parts. The strongly connected portion of the giant component, represented by the central circle, is that portion in which every vertex can be reached from every other. The two sides of the bow-tie represent (1) those vertices from which the strongly connected component can be reached but which is not possible to reach from the strongly connected component and (2) those vertices which can be reached from the strongly connected component but from which it is not possible to reach the strongly connected component. The solution of Eqs. (35) and (36) with $G_0(x)$ and $G_1(x)$ defined according to Eq. (61) gives the number of vertices, as a fraction of $N$, in the strongly connected component plus those vertices from which the strongly connected component can be reached. Using $F_0(x)$ and $F_1(x)$ (Eq. (60)) in place of $G_0(x)$ and $G_1(x)$ gives a different solution, which represents the fraction of the graph in the strongly connected component plus those vertices which can be reached from it.

**B. Simulation results**

We have performed simulations of directed graphs as a check on the results above. Generation of random directed graphs with known joint degree distribution $p_{jk}$ is somewhat more complicated that the generation of undirected graphs discussed in Section II G. The method we use is as follows. First, it is important to ensure that the average in- and out-degrees of the graph are the same, or equivalently that $p_{jk}$ satisfies Eq. (58). If this is not the case, at least to good approximation, then generation of the graph will be impossible. Next, we generate a set of $N$ in/out-degree pairs $(j_i, k_i)$, one for each vertex $i$, according to the joint distribution $p_{jk}$, and calculate the sums $\sum_i j_i$ and $\sum_i k_i$. These sums are required to be equal if there are to be no dangling edges in the graph, but in most cases we find that they are not. To rectify this we use a simple procedure. We choose a vertex $i$ at random, discard the numbers $(j_i, k_i)$ for that vertex and generate new ones from the distribution $p_{jk}$. We repeat...
this procedure until the two sums are found to be equal. Finally, we choose random in/out pairs of edges and join them together to make a directed graph. The resulting graph has the desired number of vertices and the desired joint distribution of in and out degree.

We have simulated directed graphs in which the distribution \( p_{jk} \) is given by a simple product of independent distributions of in- and out-degree. (As pointed out in Section IIIA, this is not generally the case for real-world directed graphs, where in- and out-degree may be correlated.) In Fig. 6 we show results from simulations of graphs with identically distributed (but independent) in- and out-degrees drawn from the exponential distribution, Eq. (14). For this distribution, solution of the critical-point equation \( G_1^0(1) = 1 \) shows that the giant component first appears at \( \kappa_c = [\log 2]^{-1} = 1.4427 \). The three curves in the figure show the distribution of numbers of vertices accessible from each vertex in the graph for \( \kappa = \kappa_c, 0.8, \) and 0.5. The critical distribution follows a power-law form (see Section II C), while the others show an exponential cutoff. We also show the exact distribution derived from the coefficients in the expansion of \( H_1(x) \) about zero. Once again, theory and simulation are in good agreement. A fit to the distribution for the case \( \kappa = \kappa_c \) gives a value of \( \alpha = 1.50 \pm 0.02 \), in good agreement with Eq. (47).

IV. BIPARTITE GRAPHS

The collaboration graphs of scientists, business-people, and movie actors discussed in Section I are all examples of bipartite graphs. In this section we study the theory of bipartite graphs with arbitrary degree distributions. To be concrete, we will speak in the language of “actors” and “movies,” but clearly all the developments here are applicable to academic collaborations, boards of directors, or any other bipartite graph structure.

A. Generating functions and basic results

Consider then a bipartite graph of \( M \) movies and \( N \) actors, in which each actor has appeared in an average of \( \mu \) movies and each movie has a cast of average size \( \nu \) actors. Note that only three of these parameters are independent, since the fourth is given by the equality

\[
\frac{\mu}{M} = \frac{\nu}{N}. \tag{66}
\]

Let \( p_j \) be the probability distribution of the degree of actors (i.e., of the number of movies in which they have appeared) and \( q_k \) be the distribution of degree (i.e., cast size) of movies. We define two generating functions which generate these probability distributions thus:

\[
f_0(x) = \sum_j p_j x^j, \quad g_0(x) = \sum_k q_k x^k. \tag{67}
\]

(It may be helpful to think of \( f \) as standing for “film,” in order to keep these two straight.) As before, we necessarily have

\[
f_0(1) = g_0(1) = 1, \quad f_0'(1) = \mu, \quad g_0'(1) = \nu. \tag{68}
\]

If we now choose a random edge on our bipartite graph and follow it both ways to reach the movie and actor which it connects, then the distribution of the number of other edges leaving those two vertices is generated by the equivalent of Eq. (9):

\[
f_1(x) = \frac{1}{\mu} f_0'(x), \quad g_1(x) = \frac{1}{\nu} g_0'(x). \tag{69}
\]

Now we can write the generating function for the number of co-stars (i.e., actors in shared movies) of a randomly chosen actor as

\[
G_0(x) = f_0(g_1(x)). \tag{70}
\]

If we choose a random edge, then the distribution of number of co-stars of the actor to which it leads is generated by

\[
G_1(x) = f_1(g_1(x)). \tag{71}
\]

These two functions play the same role in the one-mode network of actors as the functions of the same name did for the unipartite random graphs of Section II. Once we have calculated them, all the results from Section II follow exactly as before.

The numbers of first and second neighbors of a randomly chosen actor are

\[
z_1 = G_0'(1) = f_0'(1) g_1'(1), \tag{72}
\]

\[
z_2 = G_0'(1) G_1'(1) = f_0'(1) f_1'(1) [g_1'(1)]^2. \tag{73}
\]

Explicit expressions for these quantities can be obtained by substituting from Eqs. (67) and (69). The average vertex–vertex distance on the one-mode graph is given as before by Eq. (53). Thus, it is possible to estimate average distances on such graphs by measuring only the numbers of first and second neighbors. A word of caution here, however: because of the bipartite nature of a collaboration graph, many of the second neighbors of a vertex will themselves be neighbors of more than one first neighbor of that vertex. It is crucial that such neighbors be counted only once in the measurement of \( z_2 \). If they are counted every time they occur, Eq. (53) will give erroneous results for \( \ell \).

The distribution of the sizes of the connected components in the one-mode network is generated by Eq. (27), where \( H_1(x) \) is a solution of Eq. (26). The asymptotic results of Section II E generalize simply to the bipartite case, and the average size of a connected component (excluding the giant component, if there is one) is

\[
\langle s \rangle = 1 + \frac{G_0'(1)}{1 - G_1'(1)}. \tag{74}
\]
as before. This diverges when \(G'_1(1) = 1\), marking the first appearance of the giant component. Equivalently, the giant component first appears when

\[
\sum_{jk} jk (jk - j - k)p_j q_k = 0.
\]  

(76)

The size \(S\) of the giant component, as a fraction of the total number \(N\) of actors, is given as before by the solution of Eqs. (35) and (36).

Of course, all of these results work equally well if “actors” and “movies” are interchanged. One can calculate the average distance between movies in terms of common actors shared, the size and distribution of connected components and movies, and so forth, using the formulas given above, with only the exchange of \(f_0\) and \(f_1\) for \(g_0\) and \(g_1\). The formula (75) is, not surprisingly, invariant under this interchange, so that the position of the onset of the giant component is the same regardless of whether one is looking at actors or movies.

**B. Clustering**

Watts and Strogatz [14] have introduced the concept of clustering in social networks. Clustering is a measure of the increased propensity of pairs of people to be acquainted with one another if they have another acquaintance in common. Watts and Strogatz defined a clustering coefficient which measures the degree of clustering on a graph. For our purposes, the definition of this coefficient is

\[
C = \frac{3 \times \text{number of triangles on the graph}}{\text{number of connected triples of vertices}} = \frac{3N_{\Delta}}{N_3}.
\]

(77)

Here “triangles” are trios of vertices each of which is connected to both of the others, and “connected triples” are trios in which at least one is connected to both the others. The factor of 3 in the numerator accounts for the fact that each triangle contributes to three connected triples of vertices, one for each of its three vertices. With this factor of 3, the value of \(C\) lies strictly in the range from zero to one. In the directed and undirected unipartite random graphs of Sections II and III, \(C\) is trivially zero in the limit \(N \rightarrow \infty\). In the one-mode projections of bipartite graphs, however, both the actors and the movies can be expected to have non-zero clustering. We here treat the case for actors. The case for movies is easily derived by swapping \(f_0\) and \(g_0\).

An actor who has \(z \equiv z_1\) co-stars in total contributes \(\frac{1}{2} z(z - 1)\) connected triples to \(N_3\), so that

\[
N_3 = \frac{1}{2} N \sum_z z(z - 1) r_z,
\]

(78)

where \(r_z\) is the probability of having \(z\) co-stars. As shown above (Eq. (70)), the distribution \(r_z\) is generated by \(G_0(x)\) and so

\[
N_3 = \frac{1}{2} NG_0''(1).
\]

(79)

A movie which stars \(k\) actors contributes \(\frac{1}{6} k(k-1)(k-2)\) triangles to the total triangle count in the one-mode graph. Thus the total number of triangles on the graph is the sum of \(\frac{1}{6} k(k-1)(k-2)\) over all movies, which is given by

\[
N_{\Delta} = \frac{1}{6} M \sum_k k(k-1)(k-2)p_k = \frac{1}{6} Mg_0''(1).
\]

(80)

Substituting into Eq. (77), we then get

\[
C = \frac{M g_0''(1)}{N G_0''(1)}.
\]

(81)

Making use of Eqs. (66), (67), and (70), this can also be written as

\[
\frac{1}{C} - 1 = \frac{\mu_2 - \mu_1}{\mu_1 \nu_1 (2\nu_1 - 3\nu_2 + \nu_3)}.
\]

(82)

where \(\mu_n = \sum_k k^n p_k\) is the \(n\)th moment of the distribution of numbers of movies in which actors have appeared, and \(\nu_n\) is the same for cast size (number of actors in a movie).

**C. Example**

To give an example, consider a random bipartite graph with Poisson-distributed numbers of both movies per actor and actors per movie. In this case, following the derivation of Eq. (12), we find that

\[
f_0(x) = e^{\nu(x-1)}, \quad g_0(x) = e^{\nu(x-1)},
\]

(83)

and \(f_1(x) = f_0(x)\) and \(g_1(x) = g_0(x)\). Thus

\[
G_0(x) = G_1(x) = e^{\nu(e^{\nu-1})-1}.
\]

(84)

This implies that \(z_1 = \mu \nu\) and \(z_2 = (\mu \nu)^2\), so that

\[
\ell = \frac{\log N}{\log \mu \nu} = \frac{\log N}{\log z},
\]

(85)

just as in a normal Poisson-distributed random graph. The average size \(\langle s \rangle\) of a connected component of actors from Eq. (74) is

\[
\langle s \rangle = \frac{1}{1 - \mu \nu},
\]

(86)
which diverges, yielding a giant component, at $\mu \nu = z = 1$, also as in the normal random graph. From Eqs. (35) and (36), the size $S$ of the giant component as a fraction of $N$ is a solution of

$$S = 1 - e^{\mu e^{-\nu S} - 1}. \quad (87)$$

And from Eq. (81), the clustering coefficient for the one-mode network of actors is

$$C = \frac{M \nu^3}{N \nu^2 (\mu^2 + \mu)} = \frac{1}{\mu + 1}, \quad (88)$$

where we have made use of Eq. (66).

Another quantity of interest is the distribution of numbers of co-stars, i.e., of the numbers of people with whom each actor has appeared in a movie. As discussed above, this distribution is generated by the function $G(x)$ defined in Eq. (70). For the case of the Poisson degree distribution, we can perform the derivatives, Eq. (4), and setting $x = 0$ we find that the probability $r_z$ of having appeared with a total of exactly $z$ co-stars is

$$r_z = \frac{\mu^z}{z!} e^{\mu(e^{-\nu} - 1)} \sum_{k=1}^{z} \left\{ z \atop k \right\} [\mu e^{-\nu}]^k, \quad (89)$$

where the coefficients $\left\{ z \atop k \right\}$ are the Stirling numbers of the second kind [35]

$$\left\{ z \atop k \right\} = \sum_{r=1}^{k} \frac{(-1)^{k-r}}{r!(k-r)!} r^z. \quad (90)$$

D. Simulation results

Random bipartite graphs can be generated using an algorithm similar to the one described in Section III B for directed graphs. After making sure that the required degree distributions for both actor and movie vertices have means consistent with the required total numbers of actors and movies according to Eq. (66), we generate vertex degrees for each actor and movie at random and calculate their sum. If these sums are unequal, we discard the degree of one actor and one movie, chosen at random, and replace them with new degrees drawn from the relevant distributions. We repeat this process until the total actor and movie degrees are equal. Then we join vertices up in pairs.

In Fig. 7 we show the results of such a simulation for a bipartite random graph with Poisson degree distribution. (In fact, for the particular case of the Poisson distribution, the graph can be generated simply by joining up actors and movies at random, without regard for individual vertex degrees.) The figure shows the distribution of the numbers of co-stars of each actor, along with the analytic solution, Eqs. (89) and (90). Once more, the two are in good agreement.

![FIG. 7. The frequency distribution of numbers of co-stars of an actor in a bipartite graph with $\mu = 1.5$ and $\nu = 15$. The points are simulation results for $M = 10000$ and $N = 100000$. The line is the exact solution, Eqs. (89) and (90). The error bars on the numerical results are smaller than the points.](image)

V. APPLICATIONS TO REAL-WORLD NETWORKS

In this section we construct random graph models of two types of real-world networks, namely collaboration graphs and the world-wide web, using the results of Sections III and IV to incorporate realistic degree distributions into the models. As we will show, the results are in the former case in reasonably good agreement with empirical data, while in the latter case there is a significant discrepancy between theory and data, possibly indicating the presence of some social phenomena that are not incorporated in a random graph.

A. Collaboration networks

In this section we construct random bipartite graph models of the known collaboration networks of businesspeople [23,24], movie actors [14], and scientists [26]. As we will see, the random graph works well as a model of these networks, giving good order-of-magnitude estimates of all quantities investigated, and in some cases giving results of startling accuracy.

Our first example is the collaboration network of the members of the boards of directors of the Fortune 1000 companies (the one thousand US companies with the highest revenues). The data come from the 1999 Fortune 1000 [23,24] and in fact include only 914 of the 1000, since data on the boards of the remaining 86 were not available. The data form a bipartite graph in which one type of vertex represents the boards of directors, and the
other type the members of those boards, with edges connecting boards to their members. In Fig. 8 we show the frequency distribution of the numbers of boards on which each member sits, and the numbers of members of each board. As we see, the former distribution is close to exponential, with the majority of directors sitting on only one board, while the latter is strongly peaked around 10, indicating that most boards have about 10 members.

Using these distributions, we can define generating functions \( f_0(x) \) and \( g_0(x) \) as in Eq. (24), and hence find the generating functions \( G_0(x) \) and \( G_1(x) \) for the distributions of numbers of co-workers of the directors. We have used these generating functions and Eqs. (72) and (81) to calculate the expected clustering coefficient \( C \) and the average number of co-workers \( z \) in the one-mode projection of board directors on a random bipartite graph with the same vertex degree distributions as the original dataset. In Table I we show the results of these calculations, along with the same quantities for the real Fortune 1000. As the table shows the two are in remarkable—almost perfect—agreement.

In fact it is not just the average value of \( z \) that we can calculate from our generating functions, but the entire distribution: since the generating functions are finite polynomials in this case, we can simply perform the derivatives to get the probability distribution \( r_z \). In Fig. 9, we show the results of this calculation for the Fortune 1000 graph. The points in the figure show the actual distribution of \( z \) for the real-world data, while the solid line shows the theoretical results. Again the agreement is excellent. The dashed line in the figure shows the distribution for an ordinary Poisson random graph with the same mean. Clearly this is a significantly inferior fit.

These results appear encouraging. We have been able to formulate a model which takes as its input the distributions of the sizes of boards of directors and the number of boards that each directors sits on, and which predicts with good accuracy such things as the amount of clustering in the corresponding one-mode network and the distributions of numbers of co-workers of the board directors. Conversely, one could argue that what we have really shown is that the statistical properties studied here for the network of boards of directors do not show any obvious signature of an underlying social dynamics. Rather, the statistical properties of the network can be accurately predicted by a model which assumes the network is just a random graph, albeit a special kind of random graph which takes some features of the real-world network into account. This does not imply that no interesting social phenomena are to be found the directors network—certainly it is not true that people choose to sit on boards at random, or that boards select their members randomly—but these forces appear to have little influence on the coarse structural features of the net-
work. As a contrast to this situation, we now look at three other graphs for which our theory also works well, but for which there are noticeable deviations from the random graph predictions.

We consider the graph of movie actors and the movies in which they appear [14] and graphs of scientists and the papers they write in physics and biomedical research [26]. In Table I we show results for the clustering coefficients and average coordination numbers of the one-mode projections of these graphs onto the actors or scientists. As the table shows, our theory gives results for these figures which are of the right general order of magnitude, but typically deviate from the empirically measured figures by a factor of two or so. In the insets of Fig. 9 we show the distributions of numbers of collaborations in the movie actor and physics graphs, and again the match between theory and real data is good, but not as good as with the Fortune 1000.

The figures for clustering and mean numbers of collaborators are particularly revealing. The former is uniformly about twice as high in real life as our model predicts for the actor and scientist networks. This shows that there is a significant tendency to clustering in these networks, in addition to the trivial clustering one expects on account of the bipartite structure. This may indicate, for example, that scientists tend to introduce pairs of their collaborators to one another, thereby encouraging clusters of collaboration. The figures for average numbers of collaborators show less deviation from theory than the clustering coefficients, but nonetheless there is a clear tendency for the numbers of collaborators to be smaller in the real-world data than in the models. This indicates that scientists and actors tend to collaborate repeatedly with the same people, thereby reducing their total number of collaborators below the number that would naively be expected if we consider only the numbers of papers that they write. It would certainly be possible to take sociological effects such as these into account in a more sophisticated model of collaboration practices.

B. The world-wide web

Having seen one example for which the assumptions of the random graph work quite well, we now turn to one for which they do not—the web. As we will see, it seems to be difficult to make a model of the web as a random graph which agrees well with the observed properties of the web. This may indicate that there is non-trivial structure in the web which is not present in the random graph, although other explanations are also possible, as discussed below.

As we pointed out in Section III A, it is not at present possible to make an very accurate random-graph model of the web, because to do so we need to know the joint distribution $p_{jk}$ of in- and out-degrees of vertices, which has not to our knowledge been measured. However, we can make a simple model of the web by assuming in- and out-degree to be independently distributed according to their known distributions. Equivalently, we assume that the joint probability distribution factors according to $p_{jk} = p_j q_k$.

Broder et al. [21] give results showing that the in- and out-degree distributions of the web are approximately power-law with exponents $\tau_{\text{in}} = 2.1$ and $\tau_{\text{out}} = 2.7$, although there is some deviation from the perfect power-law form for small degree, as shown in Fig. 10. We find the degree distributions to be well-fitted by the form

$$p_k = \begin{cases} p_0 & \text{for } k = 0 \\ C(k + k_0)^{-\tau} & \text{for } k \geq 1 \end{cases}$$

where the constant $C$ is fixed by the requirement of normalization. The values of the constants $k_0$ and $\tau$ are found by least-squares fits to the data, giving values of 1.25 and 6.94 for $k_0$ and 2.10 and 2.82 for $\tau$, for the in- and out-degree distributions respectively. With these choices, the data and Eq. (91) match closely (see Fig. 10 again). The fraction $p_0$ of vertices with in- or out-degree zero is difficult to measure empirically [36]. Vertices with in-degree zero are particularly difficult to deal with because data in studies of the web, such as Refs. [18] and [21], are derived from “crawls” in which web pages are found by following links from one to another. Clearly no web pages with zero in-degree are ever found in such a crawl. So instead we take another approach. We choose the value of $p_0$ for in- and out-degree distributions in order to satisfy Eq. (58), which leaves us with a one-parameter family of web models. We can choose the one parameter in order to optimize our fit to the observed properties of the web.
values quoted above we find to be in- and out-degree distributions, which for the parameter $\tau \geq 2$ is

$$F_0(x) = F_1(x) = p_0 + (1 - p_0) \frac{x \Phi_{\tau}(x, k_0 + 1)}{\zeta(\tau, k_0 + 1)}, \quad (92)$$

and similarly for $G_0$ and $G_1$, where $\Phi_{\tau}(x, y)$ is the Lerch $\Phi$-function and $\zeta(x, y)$ is the generalized $\zeta$-function [35]. The equalities $F_0 = F_1$ and $G_0 = G_1$ are general properties of all directed graphs for which $p_{jk} = p_j q_k$ as above. They arise from the fact that in such graphs in- and out-degree are uncorrelated, and therefore the distribution of the out-degree of a vertex does not depend on whether you arrived at it by choosing a vertex at random, or by following a randomly chosen edge.

The average in-degree of a vertex (well-defined only for $\tau \geq 2$) is

$$\langle j \rangle = F_0'(1) = (1 - p_0) \left[ 1 + \frac{\Phi_{\tau}'(1, k_0 + 1)}{\zeta(\tau, k_0 + 1)} \right], \quad (93)$$

where the prime on $\Phi$ indicates differentiation with respect to the first argument. A similar expression gives the average out-degree $\langle k \rangle$. Equating the two then gives us a single linear constraint on the values of $p_0$ for the in- and out-degree distributions, which for the parameter values quoted above we find to be

$$p_0^{(\text{in})} = 0.455 + 0.545 p_0^{(\text{out})}, \quad (94)$$

where $p_0^{(\text{in})}$ and $p_0^{(\text{out})}$ are the values of $p_0$ for the two distributions. Note that $p_0^{(\text{in})} = p_0^{(\text{out})} = 1$ is a solution of this constraint, as we would expect.

We can now use our generating functions to make estimates of various properties of the web. For example, the fraction $S$ of the graph filled by the strongly connected component plus those sites from which the strongly connected component can be reached is a solution of

$$S = 1 - G_0(1 - S). \quad (95)$$

Numerically iterating this equation, we can find $S$ as a function of $p_0^{(\text{out})}$ from zero to one. We do the same also for $F_0(x)$, giving the size of the strongly connected component plus those sites that are reachable from it. We show the results in Fig. 11. We notice the following points about this figure:

1. When $p_0^{(\text{out})} = 1$ there is no giant component because there are no links out of any site.
2. When $p_0^{(\text{out})} = 0$ every site has at least one link departing from it, and this means that the strongly connected component can (almost surely) be reached from any site. As a result the graph consists only of those sites in the strongly connected component and those from which the strongly connected component can be reached. This implies that the value of the dashed curve at this point is the size of the strongly connected component, which is about 33%.
3. The size of the strongly connected component can be at most equal to the value given by the solid line. Thus the difference between this value and that given by the dashed line must be made up with sites from which the strongly connected component can be reached, and this constitutes a significant fraction of the graph—more than the size of the strongly connected component. This is not in agreement with what we see in the real world data [21], in which it is found that the strongly connected component is the largest part of the giant component and the two “loops” of the bow-tie are a little smaller, but roughly equal in size to one another.

What does this mean for our model? The most satisfying conclusion would be that this reveals that the web is not just a random graph but has significant underlying structure, perhaps due to social or geographical effects which are not incorporated in the model. And this may indeed be the case. However, there are other possible explanations.

One possibility is that the data we have for the web do not reflect very accurately its true structure. The observed distributions of in- and out-degree on the web require that there be a significant number of vertices with in-degree zero. Under the assumption made here of independence of in- and out-degrees of vertices, this number is never less than 45% of the entire graph (see Eq. (94)). But very few such vertices are recorded by web crawls,
VI. CONCLUSIONS

In this paper we have studied in detail the theory of random graphs with arbitrary distributions of vertex degree, including directed and bipartite graphs. We have shown how, using the mathematics of generating functions, one can calculate exactly many of the statistical properties of such graphs in the limit of large numbers of vertices. Amongst other things, we have given explicit formulas for the position of the phase transition at which a giant component forms, the size of the giant component, the average and distribution of the sizes of the other components, the average numbers of vertices a certain distance from a given vertex, the clustering coefficient, and the typical vertex–vertex distance on a graph. We have given examples of the application of our theory to the modeling of collaboration graphs, which are inherently bipartite, and the world-wide web, which is directed. We have shown that the random graph theory gives good order-of-magnitude estimates of the properties of known collaboration graphs of business-people, scientists, and movie actors, and for the first of these essentially predicts the properties of the graph exactly. For the web we find a different story: there are substantial quantitative differences between the predictions of the random graph theory and the properties of the real world-wide web. Although these differences may be attributable to incompleteness of the available data, they may also be indicative of interesting social structures in the web which are not captured by a simple random graph model.

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